Journal of Statistical Physics, Vol. 92, Nos. 1/2, 1998

Large-Deviation Principle for One-Dimensional Vector Spin Models with Kac Potentials

Paolo Buttà[1,2] and Pierre Picco[3]

Received November 19, 1998

We consider the one-dimensional planar rotator and classical Heisenberg models with a ferromagnetic Kac potential $J_\gamma(r) = \gamma J(\gamma r)$, J with compact support. Below the Lebowitz–Penrose critical temperature the limit (mean-field) theory exhibits a phase transition with a continuum of equilibrium states, indexed by the magnetization vectors $m_\beta s$, s any unit vector and m_β the Curie–Weiss spontaneous magnetization. We prove a large-deviation principle for the associated Gibbs measures. Then we study the system in the limit $\gamma \downarrow 0$ below the above critical temperature. We prove that the norm of the empirical spin average in blocks of order γ^{-1} converges to m_β, uniformly in intervals of order γ^{-p}, for any $p \geqslant 1$. We also give a lower bound to the scale on which the change of phase occurs, by showing that the empirical spin average is approximately constant on intervals having length of order $\gamma^{-1-\lambda}$ with $\lambda \in (0, 1)$ small enough.

KEY WORDS: Large deviations; Kac potentials; spin vector models.

1. INTRODUCTION

Mean field models are very important to explain in a simple way the general phenomenon of phase transitions. However they have some unphysical properties as the non convexity of the canonical free energy. In the lattice gas language, this non convexity produces the totally unphysical effect that in some region of the parameters the pressure is a decreasing function of the density. Exactly the same problem appears in the famous van der Waals equation of state that comes front a molecular theory of the vapor-liquid transition. To avoid this unphysical feature of the van der

[1] INRIA CMI, Université de Provence, 39 F-13453 Marseille Cedex 13, France.
[2] Present address: Center for Mathematical Sciences Research, Rutgers, the State University of New Jersey, Piscataway, New jersey 08854-8019.
[3] CPT-CNRS, Luminy, Case 907, F-13288 Marseille Cedex 9, France.

Waals equation of state, Maxwell, in the middle of the last century, invented his equal area modification of this equation. In term of thermodynamic potentials this corresponds to take the *convex* envelope of the mean field canonical free energy. In the end of the fifties, M. Kac, G. Uhlenbeck, & P. C. Hemmer,[24, 25, 26] found a microscopic model for the vapor-liquid transition that explains in a very clear way the Maxwell modification of the van der Waals theory. More precisely they derived the phase diagram of the van der Waals theory, comprehensive of the Maxwell rule, from the thermodynamics of a system with interaction $\gamma J(\gamma r)$, $J(r) = \exp[-|r|]$, in the limit $\gamma \downarrow 0$. These results were extended by J. Lebowitz & O. Penrose,[29] to a larger class of potentials. There is a review by P. Hemmer & J. Lebowitz,[22] appeared in the middle of the seventies, where more complete references can be found.

As well known, mean field theories present some other unsatisfactory features, e.g. phase transitions occur independently on the dimension in the space. Then the behavior of a system with long but finite range (i.e., with γ small but positive) can be very different from the one of the corresponding mean field model. Therefore it is natural to wonder what kind of informations about the system, before the limit $\gamma \downarrow 0$ is taken, can be extracted from the mean field theory. Strangely enough such a basic question has not been investigated for a long time.

Only recently, after an important paper by M. Cassandro, E. Orlandi, & E. Presutti,[12] a new interest for Kac models came. In ref. 12 the authors give a very complete description of the infinite volume one dimensional Ising–Kac model: they characterize the typical configurations of the (unique) Gibbs state and the interface between two phases. Part of these results were extended by T. Bodineau,[3] to a system of bounded continuous spins, but the canonical free energy of the corresponding mean field model is still a double well one. Disordered one dimensional Kac models were studied recently, in particular by A. Bovier, V. Gayrard, & P. Picco,[6, 7] for the Kac–Hopfield model, and by T. Bodineau,[4] for a disordered ferromagnetic Kac model. After that, the study of Ising–Kac model in more than one dimension was done by various authors. The existence of a phase transition for the system before the Kac limit was proved by M. Cassandro & E. Presutti,[13] and A. Bovier & M. Zahradnik.[8] Asymptotic from above of the critical temperature and decay of correlation functions were given by M. Cassandro, R. Marra, & E. Presutti.[11] The surface tension was investigated by G. Alberti, G. Bellettini, M. Cassandro, & E. Presutti,[1] and by O. Benois, T. Bodineau, P. Buttà, & E. Presutti,[2] while a characterization of the translation invariant states was done by P. Buttà, I. Merola, & E. Presutti.[10] The problem of considering the phase diagram of a perturbation à la Kac of an Ising model was studied by

T. Bodineau & E. Presutti.[5] Dynamical aspects of the Ising–Kac model can be found in the papers by A. De Masi, E. Orlandi, E. Presutti, & L. Triolo, see ref. 14 and references therein.

Beside the Ising model, vector spin models with an internal continuous symmetry are also very important. They exhibit radically different behaviors. A typical example is given by the classical $SO(q)$ models, and, among these, the classical XY or "planar rotator" model ($q=2$), and the classical Heisenberg model ($q=3$). In one and two dimensions, if the interaction range decays at infinity fast enough, there is no breaking of the internal symmetry, this comes from the Mermin–Wegner argument and was proved by R. Dobrushin & S. Shlosman,[15] C. Pfister,[31] and J. Fröhlich & C. Pfister.[17]

For the ferromagnetic planar rotator model with short range interactions, the main feature is that there is uniqueness of the Gibbs state in one and two dimensions, as shown by J. Bricmont, J. Fontaine, & J. Landau,[9] and A. Messager, S. Miracle, & C. Pfister.[30] However in two dimensions, at least for the nearest neighbor interaction, there is the famous Berezinskii–Kosterlitz–Thouless transition,[28] where there is no spontaneous magnetization, no breakdown of the internal symmetry but the decay of two points correlation functions is exponential at high temperature and only polynomial at low temperature. The existence of this transition was proved rigorously by J. Fröhlich & T. Spencer.[20]

In three or more dimensions it was proved by J. Fröhlich, B. Simon, & T. Spencer,[19] that the ferromagnetic planar rotator and classical Heisenberg models exhibit spontaneous magnetization and symmetry breaking at sufficiently low temperature. A complete description of all extremal, translation invariant equilibrium states was done by J. Fröhlich & C. Pfister,[18] where it was also proved that the surface tension vanishes.

From what discussed above it seems quite natural to start an analysis of the Kac version of the classical XY or more generally classical $SO(q)$ models, as done for the Ising–Kac one. Moreover, a general picture on the statistical mechanics of finite range $SO(q)$ models is not well established as for the Ising model.

For what concern the Kac limit of the thermodynamic potentials for $SO(q)$ models, C. Thompson & M. Silver,[35] proved the Lebowitz–Penrose theorem for the pressure. An analogous result for the canonical free energy is missing. The proof of such a result is standard if one has good estimates for the asymptotics of the corresponding independent (i.e. non interacting) model. This is contained in our Theorem 2.2 (for $q=2, 3$) that we did not find in the literature and we believe interesting in its own.

The first step in the systematic study of Kac models is to consider the one dimensional case, where we can expect to have a rather complete description of typical configurations. That is, to extend to models with an $SO(q)$-symmetry the results of ref. 12. In this paper we consider the Kac version of the ferromagnetic classical XY and Heisenberg models ($q = 2, 3$). We prove a large deviation principle for the associated (infinite volume) Gibbs states, by giving the explicit form of the large deviation rate functional. By going beyond the large deviation behavior, we deduce also a lower bound on the scale where the typical configurations are rigid.

2. NOTATION AND RESULTS

2.1. The Model

To each site i of $\mathbb{Z}$ is attached a spin variable σ_i taking values in $\mathbb{R}^q$, $q = 2, 3$. The a priori probability distribution ν for the σ_i's is assumed to be the normalized surface measure on S^{q-1}, the unit sphere in $\mathbb{R}^q$: $\nu(d\sigma_i) \equiv |S^{q-1}|^{-1}\, \delta(|\sigma_i| - 1)\, d\sigma_i$. We denote by σ the spin configuration $\{\sigma_i; i \in \mathbb{Z}\}$ and, for any region Λ of $\mathbb{Z}$, σ_Λ is the restriction of σ to Λ. Finally we call $\mathscr{S}$, $\mathscr{S}_\Lambda$ the space of all the spin configurations on $\mathbb{Z}$, Λ respectively.

The energy of the configuration σ in a finite region Λ of $\mathbb{Z}$ and with free boundary conditions is

$$H_\gamma(\sigma_\Lambda) = -\tfrac{1}{2} \sum_{\substack{i,\, j \in \Lambda \\ i \neq j}} J_\gamma(i-j)\, \sigma_i \cdot \sigma_j \tag{2.1}$$

where "$\cdot$" denotes the Euclidean scalar product in $\mathbb{R}^q$. $J_\gamma(i)$, $i \in \mathbb{Z}$, $\gamma \in (0, 1]$, is a Kac potential: $J_\gamma(i) = \gamma J(\gamma\, |i|)$ with $J(s)$, $s \geqslant 0$, non negative and continuous in $[0, 1]$, strictly positive in $(0, 1)$ and with support in $[0, 1]$. We assume also that J has bounded derivative in $(0, 1)$ and that it is normalized so that

$$\int_{\mathbb{R}} dx\, J(|x|) = 1 \tag{2.2}$$

The typical choice is $J(s) = \mathbb{1}_{[0,1]}$, the characteristic function of $[0, 1]$. For technical convenience, at some point in the paper, we will assume J of this particular form.

The energy inclusive of the interaction with a boundary condition $\sigma_{\Lambda^c} \in \mathscr{S}_{\Lambda^c}$ is given by

$$H_\gamma(\sigma_\Lambda \mid \sigma_{\Lambda^c}) = H_\gamma(\sigma_\Lambda) - \sum_{\substack{i \in \Lambda \\ j \in \Lambda^c}} J_\gamma(i-j)\, \sigma_i \cdot \sigma_j \tag{2.3}$$

The Gibbs measure at the inverse temperature $\beta > 0$, in the finite region Λ and with free boundary conditions is the probability distribution on S_Λ defined as

$$\mu^{\Lambda}_{\beta,\gamma}(d\sigma_\Lambda) = \frac{1}{Z^{\Lambda}_{\beta,\gamma}} \exp[-\beta H_\gamma(\sigma_\Lambda)] \prod_{i \in \Lambda} \nu(d\sigma_i) \tag{2.4}$$

where $Z^{\Lambda}_{\beta,\gamma}$ is the partition function, i.e., the normalization factor that makes the r.h.s. of (2.4) into a probability measure.

Analogously we define the Gibbs measure in Λ with boundary condition σ_{Λ^c} as the probability

$$\mu^{\Lambda}_{\beta,\gamma}(d\sigma_\Lambda \mid \sigma_{\Lambda^c}) = \frac{1}{Z^{\Lambda,\sigma_{\Lambda^c}}_{\beta,\gamma}} \exp[-\beta H_\gamma(\sigma_\Lambda \mid \sigma_{\Lambda^c})] \prod_{i \in \Lambda} \nu(d\sigma_i) \tag{2.5}$$

We denote by $\mu_{\beta,\gamma}$ the infinite volume Gibbs state, i.e., the (unique) probability distribution on $\mathscr{S}$ that satisfies the DLR equations for the specifications (2.5):

$$\mu_{\beta,\gamma}(d\sigma_\Lambda \mid \sigma_{\Lambda^c}) = \mu^{\Lambda}_{\beta,\gamma}(d\sigma_\Lambda \mid \sigma_{\Lambda^c}) \quad \mu_{\beta,\gamma}-\text{a.s.}, \quad \forall \Lambda \subset\subset \mathbb{Z} \tag{2.6}$$

$\mu_{\beta,\gamma}$ is a shift invariant measure on $\mathscr{S}$ and can be obtained as the weak limit of the free boundary states:

$$\mu_{\beta,\gamma} = \mathrm{w}-\lim_{L \to \infty} \tilde{\mu}^{\Lambda}_{\beta,\gamma}, \qquad \Lambda = [-L, L], \quad L \in \mathbb{N} \tag{2.7}$$

where $\tilde{\mu}^{\Lambda}_{\beta,\gamma}$, $\Lambda \subset\subset \mathbb{Z}$, is the cylinder measure on $\mathscr{S}$ defined by setting $\tilde{\mu}^{\Lambda}_{\beta,\gamma}(\tilde{\Gamma}) = \mu^{\Lambda}_{\beta,\gamma}(\Gamma)$ if $\tilde{\Gamma}$ is a cylinder in $\mathscr{S}$ with basis $\Gamma \subset \mathscr{S}_\Lambda$.

2.2. The Lebowitz–Penrose Limit and the Mean Field Theory

We introduce the empirical magnetization in the finite region Λ of $\mathbb{Z}$ as

$$m_\Lambda(\sigma) = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sigma_i$$

and, for any $m \in \mathbb{R}^q$, $|m| \leqslant 1$, the canonical partition function

$$Z^{\Lambda,\sigma_{\Lambda^c}}_{\beta,\gamma}(m) = \int \prod_{i \in \Lambda} \nu(d\sigma_i) \exp[-\beta H_\gamma(\sigma_\Lambda \mid \sigma_{\Lambda^c})] \, \delta(m_\Lambda(\sigma) - m) \tag{2.8}$$

Then the quantity

$$F_\gamma(\beta, m) = \lim_{\Lambda \nearrow \mathbb{Z}} -\frac{1}{\beta|\Lambda|} \log Z_{\beta,\gamma}^{\Lambda,\sigma_{\Lambda^c}}(m) \tag{2.9}$$

is well defined, it does not depend on the boundary condition σ_{Λ^c} and it has to be interpreted as the Gibbs free energy of the macroscopic system, see, e.g., ref. 21.

The Lebowitz–Penrose theorem relates the limit free energy as $\gamma \downarrow 0$ to the corresponding one predicted by the van der Waals (mean field) theory of phase transition, comprehensive of the Maxwell rule. In our context:

Theorem 2.1. Let $F_\gamma(\beta, m)$ be as in (2.9). Then

$$\lim_{\gamma \downarrow 0} F_\gamma(\beta, m) = \mathrm{CE}(f_\beta(m)) \tag{2.10}$$

where $\mathrm{CE}(f)$ denotes the convex envelope of the function f, while $f_\beta(m)$ is the free energy of the corresponding "mean field" model, i.e.,

$$f_\beta(m) = -\frac{|m|^2}{2} + \beta^{-1} I(m) \tag{2.11}$$

where $I(m)$ denotes the entropy function of the a priori measure ν.

The entropy function $I(m)$ for our model can be computed as follows. We introduce the moment generating function

$$\phi(h) \equiv \int \nu(dv)\, e^{h \cdot v}, \qquad h \in \mathbb{R}^q \tag{2.12}$$

and we define

$$I(m) \equiv \sup_{h \in \mathbb{R}^q} \{h \cdot m - \log \phi(h)\}, \qquad m \in \mathbb{R}^q \tag{2.13}$$

By symmetry $\phi(h)$ is a function of $|h|$ only and, by using polar coordinates, one easily computes

$$\phi(h) = \hat{\phi}(|h|) \equiv \begin{cases} \mathscr{I}_0(i|h|) & \text{if } q = 2 \\ |h|^{-1} \sinh(|h|) & \text{if } q = 3 \end{cases} \tag{2.14}$$

where $\mathscr{I}_0(\cdot)$ is the Bessel function of order 0.[36] Clearly also the entropy function (2.13) depends only on the norm of its argument:

$$I(m) = \hat{I}(|m|) \equiv \sup_{t \geqslant 0} \{t\,|m| - \log \hat{\phi}(t)\} \tag{2.15}$$

We note that $\hat{\phi}(t)$, $t \in \mathbb{R}$, is a real smooth, even, strictly convex function with

$$\hat{\phi}(0) = 1, \qquad \lim_{t \to +\infty} e^{-t}\hat{\phi}(t) = 1$$

Moreover $\log \hat{\phi}(t)$, $t \in \mathbb{R}$, is a non negative, even, strictly convex function with

$$\log \hat{\phi}(0) = 0, \qquad \lim_{t \to +\infty} t^{-1} \log \hat{\phi}(t) = 1$$

Then

$$I(m) = \begin{cases} h^* \cdot m - \log \phi(h^*) & \text{if} \quad |m| < 1 \\ +\infty & \text{if} \quad |m| > 1 \end{cases} \tag{2.16}$$

with $h^* = h^*(m) = (t^*/|m|)\, m$, $|m| < 1$ where $t^* = t^*(|m|)$ is the (unique) positive number for which

$$\hat{I}(|m|) = t^* \, |m| - \log \hat{\phi}(t^*) \tag{2.17}$$

(clearly $h^* = 0$ when $m = 0$). Finally, for any $m \in S^{q-1}$, $I(m)$ is finite but the supremum is not achieved in $\mathbb{R}^q$.

The result stated in Theorem 2.1 does not depend on the lattice dimension (we have considered the case $d = 1$ to simplify notation) and it exhibits a phase transition of mean field type. In fact there is a positive solution m_β of the equation

$$\beta m_\beta = \hat{I}'(m_\beta) \tag{2.18}$$

if

$$\beta > \hat{I}''(0) = \frac{\hat{\phi}(0)}{\hat{\phi}''(0)} = q$$

and any magnetization m on the sphere of radius m_β minimizes the free energy function $f_\beta(m)$. The inverse (mean field) critical temperature is thus $\beta_c^{\mathrm{mf}} = q$. A more detailed analysis of the mean field equation can be found in Kesten & Schonmann,[27] where the mean field theory for the Heisenberg model is obtained in the limit of infinite space dimensionality.

To our knowledge there is no proof of Theorem 2.1 in the literature. Thompson *et al.*,[35] by working with the grand canonical partition function, prove the analogous statement for the thermodynamic pressure

$p_\gamma(\beta, h)$, by showing that it converges to the Legendre transform of the r.h.s. of (2.10). As already discussed in the Introduction, to prove the "canonical" version (2.10) of the Lebowitz–Penrose theorem one needs estimates on the asymptotics of the empirical average with respect to the independent model. We did not find such a result in the literatures and also we will need in the sequel more refined estimates. These are the content of the following theorem that we will prove in Section 5.

Theorem 2.2. Let $\sigma = \{\sigma_i; i \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with values in $\mathbb{R}^q$, $q = 2, 3$, and probability distribution $\nu(d\sigma_i) \equiv |S^{q-1}|^{-1} \delta(|\sigma_i| - 1)\, d\sigma_i$. Let ν_N be the probability distribution in $\mathbb{R}^q$ of the empirical average $m_N(\sigma) = N^{-1} \sum_{i=1}^N \sigma_i$. Then, for any $N \geqslant 2$, ν_N is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^q$ and its density is identically 0 for $|m| > 1$. Moreover, when $|m| < 1$, the function

$$\varepsilon(m, N) \equiv \frac{1}{N} \log \frac{d\nu_N}{dm}(m) + I(m) \tag{2.19}$$

has the following property. For any $r \in (0, 1)$ there is $c(r) > 0$ such that

$$|\varepsilon(m, N)| \leqslant c(r) \frac{\log N}{N} \qquad \forall m \in \mathbb{R}^q : |m| \leqslant r \tag{2.20}$$

Finally, there is a constant $b > 0$ such that, for any $r \in (0, 1)$,

$$\nu_N(|m| > r) \leqslant (b(1-r))^{N/12} \tag{2.21}$$

With this result the proof of Theorem 2.1 is quite standard,[29] we omit it to make shorter the paper.

2.3. Block Spins and the Continuum Approximation

The phase transition described above is due to a mean field effect and, as already noticed, it occurs also in lattice dimension 1 and 2, when the system has an unique infinite volume state at any $\gamma > 0$. Here we consider the one dimensional case and we analyze the typical behavior of the system in the limit $\gamma \downarrow 0$. Since the relevant scale of the system is the interaction range γ^{-1} that diverges as $\gamma \downarrow 0$, to see a non trivial structure we have to look at the collective behavior of the system. This suggests that the relevant quantities are appropriate empirical averages of spins. Moreover, following,[12] it is convenient to scale distances by γ and to work directly on the continuum. With this in mind we introduce the following definitions.

We denote by $\mathcal{M}$ the space of all measurable functions $m: \mathbb{R} \to \mathbb{R}^q$ such that $|m(x)| \leqslant 1$ for any $x \in \mathbb{R}$. $\mathcal{M}$ is equipped with the weak L_2-loc topology with respect to which it is compact and convex. For any $\delta > 0$ we denote by $\mathcal{D}^{(\delta)}$ the partition of $\mathbb{R}$ into intervals $[n\delta, (n+1)\delta)$, $n \in \mathbb{Z}$. The "coarse graining transformation" $m \mapsto m^{(\delta)}$ of $\mathcal{M}$ into itself is defined by setting

$$m^{(\delta)}(x) = \frac{1}{\delta} \int_{n\delta}^{(n+1)\delta} dy\, m(y) \qquad \text{when} \quad x \in [n\delta, (n+1)\delta) \tag{2.22}$$

For each $\gamma > 0$ we define the continuous injective map $\sigma \mapsto \sigma_\gamma$ of $\mathcal{S}$ into $\mathcal{M}$ by setting

$$\sigma_\gamma(x) = \sigma_i \qquad \text{when} \quad x \in [i\gamma, (i+1)\gamma) \tag{2.23}$$

Clearly σ_γ is a $\mathcal{D}^{(\gamma)}$-measurable function and we call "block spin transformation" of size $\delta > \gamma$ the map $\sigma \mapsto \sigma_\gamma^{(\delta)}$ of $\mathcal{S}$ into $\mathcal{M}$. In the sequel, to simplify notations, we assume that the Kac parameter γ and any coarse graining parameter δ we introduce belong to the set $\{2^{-n}; n \in \mathbb{N}\}$.

With an abuse of notation, we denote by the same symbol $\mu_{\beta,\gamma}$ the image of the Gibbs measure $\mu_{\beta,\gamma}$ on $\mathcal{M}$ via the map (2.23). That is, for any measurable set A in $\mathcal{M}$, we shorthand $\mu_{\beta,\gamma}(A) = \mu_{\beta,\gamma}(\{\sigma \in \mathcal{S} : \sigma_\gamma \in A\})$.

2.4. Results

We will restrict ourself to the case $\beta > \beta_c^{\mathrm{mf}}$ since this is the more interesting one. When $\beta \leqslant \beta_c^{\mathrm{mf}}$ the picture is quite trivial, since one can prove that for any $\delta > 0$ the marginal of $\mu_{\beta,\gamma}$ on the block spins $\sigma_\gamma^{(\delta)}$ gives full measure to the profile $m^{(\delta)} \equiv 0$ as $\gamma \downarrow 0$.

Our first result describes the effect of the Lebowitz–Penrose phase transition in the structure of $\mu_{\beta,\gamma}$ for $\gamma \downarrow 0$. Given $m \in \mathbb{R}^q$, $|m| < 1$, we denote by $\nu_m^{\mathcal{S}}$ the product measure on $\mathcal{S}$ such that, for any $i \in \mathbb{Z}$, $\nu_m^{\mathcal{S}}(d\sigma_i) = \phi(h^*)^{-1} \exp[h^* \cdot \sigma_i]\, \nu(d\sigma_i)$ where $h^* \in \mathbb{R}^q$ is chosen such that $\nu_m(\sigma_i) = m$ (see also Section 5, formula (5.5)). Then

Theorem 2.3. Let $\beta > \beta_c^{\mathrm{mf}} = q$ and m_β as in (2.18). Then, for any $\delta > 0$, $\zeta \in (0, m_\beta)$ and $p > 0$,

$$\lim_{\gamma \downarrow 0} \mu_{\beta,\gamma}(\{\, \big| |m^{(\delta)}(x)| - m_\beta \big| \leqslant \zeta \text{ for any } |x| \leqslant \gamma^{-p}\}) = 1 \tag{2.24}$$

Moreover

$$\mathrm{w}-\lim_{\gamma \downarrow 0} \mu_{\beta,\gamma} = \int \nu(ds)\, \nu_{m_\beta s}^{\mathcal{S}} \tag{2.25}$$

Theorem 2.3 shows that the block spins $\sigma_\gamma^{(\delta)}$ are close to the mean field spontaneous magnetizations $\{m_\beta s; s\in S^{Q-1}\}$.

In fact we can prove a stronger result where we allow the parameters $\delta\equiv\delta(\gamma)\downarrow 0$, $\zeta\equiv\zeta(\gamma)\downarrow 0$, as $\gamma\downarrow 0$ and get a localization in the sphere of radius m_β for a length which is exponential in a power of γ^{-1}.

Theorem 2.4. Let $\beta>\beta_c^{\mathrm{mf}}=q$ and m_β as in (2.18). Then there exists an absolute constants $c>0$ such that if $\delta''=\delta''(\gamma)$ and $\zeta''=\zeta''(\gamma)$ satisfy $\delta''(\zeta'')^3\geqslant 3c\gamma^\alpha$ with $0<\alpha<(2(6+q))^{-1}<1$ then, for all $\epsilon>0$,

$$\mu_{\beta,\gamma}(\{\,|\,|m^{(\delta'')}(x)|-m_\beta|\leqslant\zeta'' \text{ for any } |x|\leqslant e^{c\gamma^{-1+\alpha(1-\epsilon)}}\})\geqslant 1-e^{-\epsilon c\gamma^{-1+\alpha}} \tag{2.26}$$

The next question concerns the distribution of the $\sigma_\gamma^{(\delta)}$'s on the sphere of radius m_β, i.e., the change of phase along the lattice $\mathbb{Z}$. We prove that on a *macroscopic* scale which is diverging when $\gamma\downarrow 0$, typically the profiles are rigid. This gives a lower bound on the rigidity length.

Theorem 2.5. Let $\beta>\beta_c^{\mathrm{mf}}=q$ and m_β as in (2.18). For any given $0<\rho<1$ and $\Theta>0$, let

$$\mathscr{R}_L(\Theta,\rho)\equiv\{m\in\mathscr{m}: |m^{(\rho)}(x)-m^{(\rho)}(y)|\leqslant\Theta\ \forall x,\ y\in[-L,\ +L)\} \tag{2.27}$$

then for any $\lambda>0$ small enough, for any $L\leqslant\gamma^{-\lambda}$, we have

$$\lim_{\gamma\downarrow 0}\mu_{\beta,\gamma}(\mathscr{R}_L(\Theta,\rho))=1 \tag{2.28}$$

Remarks. It follows from the proof of Theorem 2.5 that the parameter λ has to be smaller than $(6(6+q))^{-1}$. A priori, we can expect that the rigidity length, which can be defined as the largest L such that (2.28) is true, is an inverse power of γ and has to be a decreasing function of q since there are more possibilities to be non rigid when q increases. Our bound has this property but we do not believe that it is optimal. Moreover, on the heuristic level, that is looking at the large deviation functional defined in Theorem 2.6 below and ignoring the error terms, for the plane rotator model, $q=2$, the cost to make a spin wave on a circle of radius m_β, that is rotating smoothly the angle, say from 0 to L, is of order $(\Delta\theta)^2(\gamma L)^{-1}$ where $\Delta\theta\equiv\theta_L-\theta_0$. Therefore we can expect that the profiles are rigid on a macroscopic length $L=o(\gamma^{-1})$ and start making spin waves on a length $L=O(\gamma^{-1})$. Analogously, in both cases $q=2$ and $q=3$, we can expect a finite deviation of the modulus of the block spins from the mean field value m_β on a length which is exponential in γ^{-1} (notice that the proof of

Theorem 2.3, Section 4, actually implies that the modulus of the block spins remains close to m_β on a length $e^{c\gamma^{-1}}$ if c is small enough).

The key ingredient to prove the above theorems is a large deviation principle for the Gibbs measures $\{\mu_{\beta,\gamma}; \gamma > 0\}$ which is the main result of our paper. To state this we need some definitions.

We introduce the "excess free energy functional" $\mathscr{F}: \mathscr{M} \to [0, \infty]$ as

$$\mathscr{F}(m) = \int dx(f_\beta(m(x)) - f_\beta(m_\beta)) + U(m) \tag{2.29}$$

where

$$U(m) = \frac{1}{4}\int dx \int dy\, J(|x-y|)\,|m(x) - m(y)|^2 \tag{2.30}$$

and $f_\beta(m)$ is defined in (2.11). In (2.30) and in the rest of the paper we shorthand $f_\beta(m_\beta s) = f_\beta(m_\beta)$ for any $s \in S^{q-1}$ (recall that $f_\beta(m)$ depends only on $|m|$). We introduce the subset of $\mathscr{M}$

$$\mathscr{M}^0 = \{m \in \mathscr{M}: \exists x \mapsto s(x) \in S^{q-1} \text{ s.t. } m - m_\beta s \in L_2(\mathbb{R}; \mathbb{R}^q),\ U(s) < +\infty\} \tag{2.31}$$

Then

Theorem 2.6. The functional $\mathscr{F}$ is lower semicontinuous on $\mathscr{M}$ and bounded on $\mathscr{M}^0$. Moreover, the family $\{\mu_{\beta,\gamma}; \gamma > 0\}$ satisfies the large deviation principle with rate function $\beta\mathscr{F}$. That is, for any closed set $F \subset \mathscr{M}$ and any open set $A \subset \mathscr{M}$,

$$\limsup_{\gamma \downarrow 0} \gamma \log \mu_{\beta,\gamma}(F) \leqslant -\inf_{m \in F} \beta\mathscr{F}(m) \tag{2.32}$$

$$\liminf_{\gamma \downarrow 0} \gamma \log \mu_{\beta,\gamma}(A) \geqslant -\inf_{m \in A} \beta\mathscr{F}(m) \tag{2.33}$$

3. LARGE DEVIATION PRINCIPLE

This section is devoted to the proof of Theorem 2.6. Since we work in the infinite volume, if we merely perform the continuum approximation everywhere, we cannot control the errors. Then, as in ref. 12, we "localize the transition to the continuum" by using the strategy proposed by Ruelle to study superstable interactions.[32, 33] we have here new difficulties with respect to Ref. 12, coming from the vector valued nature of the single spin state space. A key ingredient is Theorem 2.2 which allows us to control the entropy contribution coming from the block spin transformation. Moreover, because of the continuous symmetry of the system, we need some extra work when proving the lower semicontinuity property of the free energy functional $\mathscr{F}$. Also a more attention is required when computing the cost in $\mathscr{F}$ of "non equilibrium" configurations.

The section is divided into four subsections. In the first one we prove a preliminary lemma on the block spin approximation. The second one is devoted to the proof of the required properties of the free energy functional $\mathscr{F}$. Finally, in the third and fourth subsections we prove the upper and the lower bounds (2.32) and (2.33).

3.1. Block Spin Approximation

We start with some definitions. Let $T=[t_1, t_2]$ be a finite interval of $\mathbb{R}$. For any $m \in \mathscr{M}$ we denote by m_T its restriction on T. A cylinder set in $\mathscr{M}$ with basis in T is any subset Γ of $\mathscr{M}$ with the following property: for any $m \in \Gamma$ and $\tilde{m} \in \mathscr{M}$, $m_T(x)=\tilde{m}_T(x)$ almost surely (in T), implies $\tilde{m} \in \Gamma$. Therefore the set $\{\sigma : \sigma_\gamma \in \Gamma\}$ is a cylinder in $\mathscr{S}$ with basis in $\Delta_T \equiv \{i \in \mathbb{Z} : [\gamma^{-1} t_1] \leqslant i < [\gamma^{-1} t_2]\}$. For any finite subset Δ of $\mathbb{Z}$ containing Δ_T and any spin configuration $\sigma_{\Delta^c} \in \mathscr{S}_{\Delta^c}$ we define

$$Z_{\beta,\gamma}^{\Delta;\sigma_{\Delta^c}}(\Gamma) \equiv \int \prod_{i \in \Delta} \nu(d\sigma_i) \exp[-\beta H_\gamma(\sigma_\Delta \mid \sigma_{\Delta^c})] \mathbb{1}_\Gamma(\sigma) \tag{3.1}$$

where we shorthand with $\mathbb{1}_\Gamma$ the characteristic function of the set $\{\sigma \in \mathscr{S} : (\sigma_\gamma)_T = m$ a.s. for some $m \in \Gamma\}$. We introduce the continuum version of the Hamiltonian (2.3),

$$\begin{aligned} E(m_T \mid m_{T^c}) = & -\tfrac{1}{2} \int_T dx \int_T dy\, J(|x-y|)\, m(x) \cdot m(y) \\ & - \int_T dx \int_{T^c} dy\, J(|x-y|)\, m(x) \cdot m(y) \end{aligned} \tag{3.2}$$

Recalling the definition (2.11) and using the normalization assumption (2.2) one easily checks that

$$\begin{aligned} & E(m_T \mid m_{T^c}) + \beta^{-1} \int_T dx\, I(m(x)) \\ & \quad = \mathscr{F}_T(m_T) + W_T(m_T \mid m_{T^c}) + U_{T^c}(m_{T^c}) + f_\beta(m_\beta)\, |T| \end{aligned} \tag{3.3}$$

where

$$\mathscr{F}_T(m_T) = \int_T dx(f_\beta(m(x)) - f_\beta(m_\beta))$$
$$+ \tfrac{1}{4}\int_T dx \int_T dy\, J(|x-y|)\, |m(x)-m(y)|^2 \qquad (3.4)$$

$$W_T(m_T \mid m_{T^c}) = \tfrac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, |m(x)-m(y)|^2 \qquad (3.5)$$

and

$$U_{T^c}(m_{T^c}) = -\tfrac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, |m(y)|^2 \qquad (3.6)$$

Notice that, since J is supported on $[0,1]$, $W_T(\cdot \mid m_{T^c})$ and $U_{T^c}(m_{T^c})$ depend only on $m(y)$ for $y \in T^c$ such that $\text{dist}(y, T) \leqslant 1$.

Lemma 3.1. Let $T = [t_1, t_2]$, $t_1, t_2 \in \mathbb{Z}$, $\Delta = \{i \in \mathbb{Z} : t_1 \leqslant i\gamma < t_2\}$. Then

(i) There are constants $b_1, b_2 > 0$ such that for any $\gamma, \delta^* \in \{2^{-n}; n \in \mathbb{N}\}$, $\gamma < \delta^*$, any measurable cylinder set Γ with basis T and any boundary condition $\sigma_{\Delta^c} \in \mathscr{S}_{\Delta^c}$,

$$\begin{aligned} Z^{\Delta, \sigma_{\Delta^c}}_{\beta,\gamma}(\Gamma) \leqslant{} & (1 + 4\pi(\delta^*\gamma^{-1})^{b_2})^{|T|/\delta^*} \exp[\beta b_1 \delta^*\gamma^{-1} |T|] \\ & \times \exp[-\beta\gamma^{-1}(U_{T^c}((\sigma_\gamma^{(\delta^*)})_{T^c}) + f_\beta(m_\beta)\,|T|)] \\ & \times \exp[-\beta\gamma^{-1} \inf_{m \in \Gamma} \{\mathscr{F}_T(m_T^{(\delta^*)}) + W_T(m_T^{(\delta^*)} \mid (\sigma_\gamma^{(\delta^*)})_{T^c})\}] \end{aligned}$$

(ii) For any $m \in \mathscr{M}$ such that $\|m_T\|_\infty < 1$, $\rho \in (0, 1 - \|m_T\|_\infty)$, $\delta^* \in \{2^{-n}; n \in \mathbb{N}\}$ define

$$V_{\delta^*, \rho, T}(m) = \{\tilde{m} \in \mathscr{M} : \|\tilde{m}_T^{(\delta^*)} - m_T^{(\delta^*)}\|_\infty < \rho\} \qquad (3.8)$$

Then, for any $\gamma \in \{2^{-n}; n \in \mathbb{N}\}$, $\gamma < \delta^*$, and any boundary condition $\sigma_{\Delta^c} \in \mathscr{S}_{\Delta^c}$,

$$Z^{\Delta,\sigma_{\Delta^c}}_{\beta,\gamma}(V_{\delta^*,\rho,T}(m))$$
$$\geqslant\left(\frac{4\pi\rho^3}{3}(\delta^*\gamma^{-1})^{-c(\|m_T\|_\infty+\rho)}\right)^{|T|/\delta^*}\exp[-\beta b_1\,\delta^*\gamma^{-1}\,|T|]$$
$$\times\exp[-\beta\gamma^{-1}(U_{T^c}((\sigma^{(\delta^*)}_\gamma)_{T^c})+f_\beta(m_\beta)\,|T|)]$$
$$\times\exp[-\beta\gamma^{-1}\sup_{\tilde m\in V_{\delta^*,\rho,T}(m)}\{\mathscr{F}_T(\tilde m^{(\delta^*)}_T)+W_T(\tilde m^{(\delta^*)}_T\mid(\sigma^{(\delta^*)}_\gamma)_{T^c})\}] \tag{3.9}$$

where b_1 is the same constant as in (3.7) while $c(\cdot)$ is the function that appears in (2.20).

Proof. For any $\delta^*\in\{2^{-n};n\in\mathbb{N}\}$ we introduce the set $\mathscr{N}=\{n\in\mathbb{Z}:t_1\leqslant\delta^*n<t_2\}$. By exploiting the smoothness properties of the interaction J one easily gets, for any $\sigma\in(S^{q-1})^{\mathbb{Z}}$,

$$\Bigg|E((\sigma^{(\delta^*)}_\gamma)_T\mid(\sigma^{(\delta^*)}_\gamma)_{T^c}+\tfrac{1}{2}\sum_{n\delta^*,\,n'\delta^*\in T}\delta^*J_{\delta^*}(n,n')\,\sigma^{(\delta^*)}_\gamma(n\delta^*)\cdot\sigma^{(\delta^*)}_\gamma(n'\delta^*)$$
$$+\sum_{\substack{n\delta^*\in T\\ n'\delta^*\in T^c}}\delta^*J_{\delta^*}(n,n')\,\sigma^{(\delta^*)}_\gamma(n\delta^*)\cdot\sigma^{(\delta^*)}_\gamma(n'\delta^*)\Bigg|\leqslant b_1'\delta^*\,|T| \tag{3.10}$$

and, for $\gamma<\delta^*$,

$$\Bigg|H_\gamma(\sigma_\Lambda\mid\sigma_{\Lambda^c})+\tfrac{1}{2}\gamma^{-1}\sum_{n\delta^*,\,n'\delta^*\in T}\delta^*J_{\delta^*}(n,n')\,\sigma^{(\delta^*)}_\gamma(n\delta^*)\cdot\sigma^{(\delta^*)}_\gamma(n'\delta^*)$$
$$+\gamma^{-1}\sum_{\substack{n\delta^*\in T\\ n'\delta^*\in T^c}}\delta^*J_{\delta^*}(n,n')\,\sigma^{(\delta^*)}_\gamma(n\delta^*)\cdot\sigma^{(\delta^*)}_\gamma(n'\delta^*)\Bigg|\leqslant b_1''\delta^*\,|\Delta| \tag{3.11}$$

where b_1' and b_1'' are suitable positive numbers depending only on J. Calling $b_1=b_1'+b_1''$ and observing that $|\Delta|=\gamma^{-1}\,|T|$ we get, for any measurable set Γ in $\mathscr{M}$,

$$\exp[-\beta b_1\,\delta^*\gamma^{-1}\,|T|]\,Z_{\mathscr{N}}(\Gamma)\leqslant Z^{\Delta,\sigma_{\Delta^c}}_{\beta,\gamma}(\Gamma)$$
$$\leqslant\exp[\beta b_1\,\delta^*\gamma^{-1}\,|T|]\,Z_{\mathscr{N}}(\Gamma) \tag{3.12}$$

where

$$Z_{\mathcal{N}}(\Gamma) \equiv \int \prod_{i \in \Delta} \nu(d\sigma_i) \exp[-\beta\gamma^{-1} E((\sigma_\gamma^{(\delta^*)})_T \mid (\sigma_\gamma^{(\delta^*)})_{T^c})] \mathbb{1}_\Gamma(\sigma) \quad (3.13)$$

If Γ is $\mathscr{D}^{(\delta^*)}$-measurable, the integrand in the r.h.s. of (3.13) depends only on the block spins $\sigma_\gamma^{(\delta^*)}(n\delta^*)$, $n \in \mathcal{N}$. Then we cant integrate first on the spin configurations in each block $\Delta_n = \{i \in \Delta : n\delta^* \leqslant i\gamma < (n+1)\delta^*\}$ for fixed value of $\sigma_\gamma^{(\delta^*)}(n\delta^*) = |\Delta_n|^{-1} \sum_{i \in \Delta_n} \sigma_i$. We obtain so

$$Z_{\mathcal{N}}(\Gamma) = \int \prod_{n \in \mathcal{N}} \nu_{\delta^*\gamma^{-1}}(d\xi_n) \exp[-\beta\gamma^{-1} E(\xi^{(\delta^*)} \mid (\sigma_\gamma^{(\delta^*)})_{T^c}] \mathbb{1}_\Gamma(\xi) \quad (3.14)$$

where $\nu_{\delta^*\gamma^{-1}}$, as in Theorem 2.2, is the law of the empirical average of $\delta^*\gamma^{-1}$ i.i.d. variables with distribution ν, $\xi^{(\delta^*)}$: $T \to \mathbb{R}^q$ is defined by setting $\xi^{(\delta^*)}(x) = \xi_n$ if $x \in [n\delta^*, (n+1)\delta^*)$. Now we prove separately (3.7) and (3.9).

(i) By (3.12) we need an upper bound for $Z_{\mathcal{N}}(\Gamma)$. We can assume that Γ is $\mathscr{D}^{(\delta^*)}$-measurable. In fact, if this is not the case, we get an upper bound by replacing Γ with the bigger $\mathscr{D}^{(\delta^*)}$-measurable set

$$\Gamma^{(\delta^*)} = \{m \in \mathscr{m} : \exists \hat{m} \in \Gamma \text{ such that } \hat{m}^{(\delta^*)} = m^{(\delta^*)}\}$$

and the r.h.s. of (3.7) does not change.

Since the integration in (3.14) is over all the possible values of $\xi = \{\xi_n; n \in \mathcal{N}\}$, we insert a partition to apply separately (2.19), (2.20) and the estimate (2.21). Let $r \in (0, 1)$ be a parameter to be fixed later. We expand

$$Z_{\mathcal{N}}(\Gamma) = \sum_{X \subseteq \mathcal{N}} Z_X^{(r)}(\Gamma) \quad (3.15)$$

where the sum is over all the subsets of $\mathcal{N}$,

$$Z_X^{(r)}(\Gamma) = \int \prod_{n \in \mathcal{N}} \nu_{\delta^*\gamma^{-1}}(d\xi_n) \exp[-\beta\gamma^{-1} E(\xi^{(\delta^*)} \mid (\sigma_\gamma^{(\delta^*)})_{T^c}] \\ \times \mathbb{1}_\Gamma(\xi) \mathbb{1}_{C_X^{(r)}}(\xi) \mathbb{1}_{B_X^{(r)}}(\xi) \quad (3.16)$$

and

$$C_X^{(r)} = \{\xi \colon |\xi_n| > r \text{ for any } n \in X\}$$
$$B_X^{(r)} = \{\xi \colon |\xi_n| \leqslant r \text{ for any } n \in \mathcal{N} \setminus X\} \quad (3.17)$$

By (2.19)

$$Z_X^{(r)}(\Gamma) = \int \prod_{n \in X} \nu_{\delta^* \gamma^{-1}}(d\xi_n) \prod_{n \in \mathcal{N} \setminus X} d\xi_n \, \mathbb{1}_\Gamma(\xi) \, \mathbb{1}_{C_X^{(r)}}(\xi) \, \mathbb{1}_{B_X^{(r)}}(\xi)$$
$$\times \exp\Bigg[-\beta\gamma^{-1} E(\xi^{(\delta^*)} \mid (\sigma_\gamma^{(\delta^*)})_{T^c}) - \delta^* \gamma^{-1}$$
$$\times \sum_{n \in \mathcal{N} \setminus X} (I(\xi_n) + \varepsilon(\xi_n, \delta^* \gamma^{-1})) \Bigg] \tag{3.18}$$

by (2.20), for any $\xi \in B_X^{(r)}$,

$$\delta^* \gamma^{-1} \sum_{n \in \mathcal{N} \setminus X} \varepsilon(\xi_n, \delta^* \gamma^{-1}) \leqslant c(r) \log(\delta^* \gamma^{-1}) \, |\mathcal{N} \setminus X| \tag{3.19}$$

on the other hand, $\hat{I}(|m|)$, see (2.15), is an increasing and continuous function for $|m| \in [0, 1]$, so that, since $\nu_{\delta^* \gamma^{-1}}$ is supported on the unit closed ball in $\mathbb{R}^q$,

$$I(\xi_n) \leqslant \hat{I}(1) \quad \nu_{\delta^* \gamma^{-1}} - \text{a.s.} \tag{3.20}$$

From (3.18), (3.19) and (3.20) we get

$$Z_X^{(r)}(\Gamma) \leqslant \exp[c(r) \log(\delta^* \gamma^{-1}) \, |\mathcal{N} \setminus X| + \hat{I}(1) \, \delta^* \gamma^{-1} \, |X|]$$
$$\times \int \prod_{n \in X} \nu_{\delta^* \gamma^{-1}}(d\xi_n) \prod_{n \in \mathcal{N} \setminus X} d\xi_n \, \mathbb{1}_\Gamma(\xi) \, \mathbb{1}_{C_X^{(r)}}(\xi) \, \mathbb{1}_{B_X^{(r)}}(\xi)$$
$$\times \exp\left[-\beta\gamma^{-1} \left(E(\xi^{(\delta^*)} \mid (\sigma_\gamma^{(\delta^*)})_{T^c}) + \beta^{-1} \int_T dx \, I(\xi^{(\delta^*)}(x)) \right) \right]$$

and then

$$Z_X^{(r)}(\Gamma) \leqslant \exp[c(r) \log(\delta^* \gamma^{-1}) \, |\mathcal{N} \setminus X| + \hat{I}(1) \, \delta^* \gamma^{-1} \, |X|]$$
$$\times \exp\left[-\beta\gamma^{-1} \inf_{m \in \Gamma} \left\{ E(m_T^{(\delta^*)} \mid (\sigma_\gamma^{(\delta^*)})_{T^c}) + \beta^{-1} \int_T dx \, I(m^{(\delta^*)}(x)) \right\} \right]$$
$$\times \left(\int d\xi_1 \, \mathbb{1}_{|\xi_1| \leqslant r} \right)^{|\mathcal{N} \setminus X|} (\nu_{\delta^* \gamma^{-1}}(|\xi_1| > r))^{|X|} \tag{3.21}$$

We notice that, for $q = 2, 3$ and any $r \in (0, 1)$, $\int d\xi_1 \, \mathbb{1}_{|\xi_1| \leqslant r} \leqslant 4\pi$, while by (2.21), $\nu_{\delta^* \gamma^{-1}}(|\xi_1| > r) \leqslant \exp[-|\log(b1 - r))| \, \delta^* \gamma^{-1}/12]$. We fix then r so close to 1 that $\hat{I}(1) \leqslant |\log(b(1 - r))|/12$. Observing that $|\mathcal{N}| = (\delta^*)^{-1} \, |T|$

and recalling (3.3) we obtain (3.7) with $b_2 = c(r)$ from (3.12), (3.15) and (3.21).

(ii) By (3.12) we need a lower bound for $Z_{\mathcal{N}}(\Gamma)$ with $\Gamma = V_{\delta^*, \rho, T}(m)$. Since $V_{\delta^*, \rho, T}(m)$ is $\mathscr{D}^{(\delta^*)}$-measurable, $Z_{\mathcal{N}}(V_{\delta^*, \rho, T}(m))$ can be written as in (3.14) with, see (3.8),

$$\mathbb{1}_{V_{\delta^*, \rho, T}(m)}(\xi) = \prod_{n \in \mathcal{N}} \mathbb{1}_{|\xi_n - m^{(\delta^*)}(n\delta^*)| < \rho} \tag{3.22}$$

But from the hypothesis on m and ρ, if $\xi \in V_{\delta^*, \rho, T}(m)$ then $|\xi_n| \leqslant \|m_T\|_\infty + \rho < 1$ for any $n \in \mathcal{N}$, so that we can apply (2.19) and (2.20) getting

$$\begin{aligned} Z_{\mathcal{N}}(V_{\delta^*, \rho, T}(m)) \geqslant \exp[-c(\|m_T\|_\infty + \rho) \log(\delta^* \gamma^{-1}) \, |\mathcal{N}|] \\ \times \int \prod_{n \in \mathcal{N}} d\xi_n \prod_{n \in \mathcal{N}} \mathbb{1}_{|\xi_n - m^{(\delta^*)}(n\delta^*)| < \rho} \\ \times \exp\left[-\beta\gamma^{-1} \left(E(\xi^{(\delta^*)} \mid (\sigma_\gamma^{(\delta^*)})_{T^c}) \right.\right. \\ \left.\left. + \beta^{-1} \int_T dx \, I(\xi^{(\delta^*)}(x))\right)\right] \end{aligned}$$

and then

$$\begin{aligned} Z_{\mathcal{N}}(V_{\delta^*, \rho, T}(m)) \geqslant \exp[-c(\|m_T\|_\infty + \rho) \\ \times \log(\delta^* \gamma^{-1}) \, |\mathcal{N}|] \left(\int d\xi_1 \, \mathbb{1}_{|\xi_1| < \rho}\right)^{|\mathcal{N}|} \\ \times \exp\left[-\beta\gamma^{-1} \sup_{\tilde{m} \in V_{\delta^*, \rho, T}(m)} \left\{E(\tilde{m}_T^{(\delta^*)} \mid (\sigma_\gamma^{(\delta^*)})_{T^c}) + \beta^{-1} \right.\right. \\ \left.\left. \times \int_T dx \, I(\tilde{m}^{(\delta^*)}(x))\right\}\right] \end{aligned} \tag{3.23}$$

By noticing that, for $q = 2, 3$, $\int d\xi_1 \, \mathbb{1}_{|\xi_1| < \rho} \geqslant 4\pi\rho^2/3$ and recalling (3.3), we get (3.9) from (3.12) and (3.23). ▮

3.2. Analysis of the Free Energy Functional

In this subsection we prove the properties of the free energy functional stated in Theorem 2.6. Let $\mathscr{F}$, U be as in (2.29), (2.30) respectively and define, for any set $\Omega \subset \mathbb{R}$ and any measurable function $f: \Omega \to \mathbb{R}^q$, $q = 1, 2, 3$,

$$\|f\|_\Omega = \sqrt{\int_\Omega dx\, |f(x)|^2} \tag{3.24}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^q$. Also we shorthand $\|f\|_{\mathbb{R}} = \|f\|$. Then

Lemma 3.2. There is $c > 0$ depending only on β and J so that the following holds.

(1) For any $m \in \mathcal{M}$ such that $\mathscr{F}(m) < +\infty$ there is a measurable map $\mathbb{R} \ni x \mapsto s(x) \in S^{q-1}$ such that

$$\|m - m_\beta s\|^2 + U(s) \leqslant c\mathscr{F}(m) \tag{3.25}$$

(2) For any $m \in \mathcal{M}$ and any measurable map $\mathbb{R} \ni x \mapsto s(x) \in S^{q-1}$,

$$\mathscr{F}(m) \leqslant c(\|m - m_\beta s\|^2 + U(s)) \tag{3.26}$$

Proof. Since $f_\beta(m) - f_\beta(m_\beta)$, as a function of $|m|$, has an absolute quadratic minima at $|m| = m_\beta$, there is $c_1 > 0$ such that, for any $m \in \mathcal{M}$,

$$c_1^{-1} \|(|m| - m_\beta)\|^2 + U(m) \leqslant \mathscr{F}(m) \leqslant c_1 \|(|m| - m_\beta)\|^2 + U(m) \tag{3.27}$$

Then

(1) Let $m \in \mathcal{M}$, $\mathscr{F}(m) < +\infty$. Define the set

$$X = \{x \in \mathbb{R} : |\, |m(x)| - m_\beta| > m_\beta/2\} \tag{3.28}$$

and let $x \mapsto s(x) \in S^{q-1}$ be any measurable map such that $s(x) = |m(x)|^{-1} m(x)$ for any $x \in X^c$. By (3.27) $\|(|m| - m_\beta)\|^2 \leqslant c_1 \mathscr{F}(m) < +\infty$ so that X has finite Lebesgue measure. Moreover, from the definition of X,

$$\|m - m_\beta s\|_X^2 \leqslant (1 + m_\beta)^2 \leqslant (1 + m_\beta)^2\, |X| \leqslant 4 \left(\frac{1 + m_\beta}{m_\beta}\right)^2 \|(|m| - m_\beta)\|_X^2$$

On the other hand, from the definition of $s(x)$,

$$\|m - m_\beta s\|_{X^c}^2 = \|(|m| - m_\beta)\|_{X^c}^2$$

so that

$$\|m - m_\beta s\|^2 \leqslant c_1 \left(1 + \left(\frac{1+m_\beta}{m_\beta}\right)^2\right) \mathscr{F}(m) \tag{3.29}$$

Finally, recalling (2.2) and using the convexity of $|\cdot|^2$,

$$U(s) \leqslant \tfrac{3}{2} \|m - m_\beta s\|^2 + 3\mathscr{F}(m) \tag{3.30}$$

From (3.29) and (3.30) the bound (3.25) for a suitable $c > 0$ follows.

(2) Let m, $s \in \mathscr{M}$, $|s(x)| = 1$ for any $x \in \mathbb{R}$. By (3.27) and arguing analogously as to get (3.30), we have

$$\mathscr{F}(m) \leqslant c_1 \|(|m| - m_\beta)\|^2 + \tfrac{3}{2} \|m - m_\beta s\|^2 + 3U(s) \tag{3.31}$$

Observing that $\|(|m| - m_\beta)\| \leqslant \|m - m_\beta s\|$, (3.26) follows from (3.31). ∎

Clearly Lemma 3.2 implies that $\mathscr{F}(m)$ is finite iff $m \in \mathscr{M}^0$, see definition (2.31). We are left with the proof of the lower semicontinuity of $\mathscr{F}$. We have to show that, if $\{m_n\}$ is a sequence in $\mathscr{M}$ converging in the weak L_2-loc topology to some $m \in \mathscr{M}$, then

$$\liminf_{n \to +\infty} \mathscr{F}(m_n) \geqslant \mathscr{F}(m) \tag{3.32}$$

clearly if $\lim_{n \to +\infty} \mathscr{F}(m_n) = +\infty$ (3.32) holds, so we can assume

$$\liminf_{n \to +\infty} \mathscr{F}(m_n) = \alpha < +\infty \tag{3.33}$$

Then there is a subsequence $\{m'_n\}$ such that $\mathscr{F}(m'_n) \leqslant \alpha + 1$ for any $n \in \mathbb{N}$. By item (1) of Lemma 3.2, there is a sequence $\{s_n\}$ of measurable maps $x \mapsto s_n(x) \in S^{q-1}$ such that

$$\|m'_n - m_\beta s_n\|^2 + U(s_n) \leqslant c(\alpha + 1) \tag{3.34}$$

Since the bounded sets in $L_2(\mathbb{R}; \mathbb{R}^q)$ are compact in the weak topology, we can extract a subsequence $\{m''_n - m_\beta s'_n\}$ converging in the weak L_2 topology to some element $f \in L_2(\mathbb{R}; \mathbb{R}^q)$. Since $m_n \to m$ weakly in L_2-loc, we conclude that the subsequence $\{s'_n\}$ converges in the weak L_2-loc topology to $\tilde{m} \equiv m_\beta^{-1}(m - f) \in \mathscr{M}$. We point out that in general $\tilde{m}(x) \notin S^{q-1}$ although $s_n(x) \in S^{q-1}$ for any $n \in \mathbb{N}$. But using the bound (3.34) for $U(s_n)$ we will prove that there is $x \mapsto s(x) \in S^{q-1}$ such that $\tilde{m} - s \in L_2(\mathbb{R}; \mathbb{R}^{q-1})$. We introduce the functionals on $\mathcal{M}$:

$$U_N(m) = \tfrac{1}{2} \int_{-N}^{N} dx(|m(x)|^2 - m(x) \cdot J * m(x)) \tag{3.35}$$

$$U_N^{(1)}(m) = \tfrac{1}{2} \int_{-N}^{N} dx(1 - m(x) \cdot J * m(x)) \tag{3.36}$$

where $N \in \mathbb{N}$ and "$*$" denotes the convolution. By the assumptions on the interaction J, for any compact set K of $\mathbb{R}$, the map i_J: $L_2(K; \mathbb{R}^q) \to C(K; \mathbb{R}^q)$: $i_J(m) \equiv J * m$ is compact. Then U_N is lower semicontinuous (notice that $|\cdot|^2$ is a convex function) while $U_N^{(1)}$ is positive and continuous. Moreover, for any $x \mapsto s(x) \in S^{q-1}$, $U_N^{(1)}(s) = U_N(s) \leqslant U(s)$, so that, from (3.34), since $s'_n \to \tilde{m}$,

$$U_N(\tilde{m}) \leqslant c(\alpha + 1), \qquad U_N^{(1)}(\tilde{m}) \leqslant c(\alpha + 1) \quad \forall N \in \mathbb{N} \tag{3.37}$$

Observing that $|J * m(x)| \leqslant 1$ for any $x \in \mathbb{R}$, from (3.37) we get, for any $N \in \mathbb{N}$,

$$\begin{aligned} 4c(\alpha + 1) &\geqslant \int_{-N}^{N} dx(|\tilde{m}(x)|^2 - |\tilde{m}(x)|) + \int_{-N}^{N} dx(1 + |\tilde{m}(x)|) \\ &= \|(1 - |\tilde{m}|)\|^2_{[-N, N]} \end{aligned}$$

so that $1 - |\tilde{m}| \in L_2(\mathbb{R})$. Moreover, by the monotone convergence theorem,

$$U(\tilde{m}) = \sup_{N \in \mathbb{N}} \tfrac{1}{4} \int_{-N}^{N} dx \int dy\, J(|x - y|)\, |\tilde{m}(x) - \tilde{m}(y)|^2$$

and, by (2.2), since $|\tilde{m}(x)| \leqslant 1$,

$$\left| U_N(\tilde{m}) - \tfrac{1}{4} \int_{-N}^{N} dx \int dy\, J(|x - y|)\, |\tilde{m}(x) - \tilde{m}(y)|^2 \right| \leqslant 1$$

so that $U(\tilde{m}) \leqslant 1 + c(\alpha + 1) < +\infty$. Arguing analogously to the proof of item (1) of Lemma 3.2, there is $x \mapsto s(x) \in S^{q-1}$ such that $\tilde{m} - s \in L_2(\mathbb{R}; \mathbb{R}^q)$ and $U(s) < +\infty$. Recalling that $m - m_\beta \tilde{m} = f \in L_2(\mathbb{R}; \mathbb{R}^q)$, we get also $m - m\beta s \in L_2(\mathbb{R}; \mathbb{R}^q)$, and then $\mathcal{F}(m) < +\infty$.

Now the proof goes on very similar to the analogous one in ref. 12, but we report it for the sake of completeness. For any finite interval T of $\mathbb{R}$, we decompose

$$\mathscr{F}(m) = \mathscr{F}_{T^c}(m_{T^c}) + \mathscr{F}^0_T(m) + \tfrac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, |m(y)|^2 \tag{3.38}$$

where

$$\mathscr{F}^0_T(m) \equiv E(m_T \mid m_{T^c}) + \beta^{-1}\int_T dx\, I(m(x)) - f_\beta(m_\beta)\, |T| \tag{3.39}$$

(see (3.2), (3.4) for notation). Since $I(\cdot)$ is a convex function and $E(\cdot\mid\cdot)$ is continuous, the functional 4 is lower semicontinuous.

Let $\varepsilon > 0$. Since $\|m''_n - m_\beta s'_n\|^2 \leqslant c(\alpha+1)$ for any $n \in \mathbb{N}$ (see (3.34)), there are a subsequence $\{n_k; k \in \mathbb{N}\}$ and an interval T_ε such that

$$\|m''_{n_k} - m_\beta s'_{n_k}\|^2_{T^c_\varepsilon} \leqslant \frac{\varepsilon}{3m_\beta} \qquad \forall k \in \mathbb{N} \tag{3.40}$$

Since $\|m - m_\beta s\|^2 < +\infty$ and $\mathscr{F}(m) < +\infty$ there is also $\tilde{T}_\varepsilon$ such that

$$\|m - m_\beta s\|^2_{\tilde{T}^c_\varepsilon} \leqslant \frac{\varepsilon}{3m_\beta} \qquad \text{and} \qquad \mathscr{F}_{\tilde{T}^c_\varepsilon}(m_{\tilde{T}^c_\varepsilon}) \leqslant \frac{\varepsilon}{3} \tag{3.41}$$

Let $T = T_\varepsilon \cup \tilde{T}_\varepsilon$; from (3.38) for $\mathscr{F}(m''_{n_k})$ and the lower semicontinuity of $\mathscr{F}^0_T(m)$, recalling (3.33), we get

$$\alpha \geqslant \mathscr{F}^0_T(m) + \liminf_{k\to+\infty} \tfrac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, |m''_{n_k}(y)|^2 \tag{3.42}$$

Using (3.38) and the second inequality in (3.41), from (3.42) we have

$$\begin{aligned}\alpha \geqslant{}& \mathscr{F}(m) - \frac{\varepsilon}{3} - \frac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, |m(y)|^2 \\ &+ \liminf_{k\to+\infty} \frac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, |m''_{n_k}(y)|^2\end{aligned} \tag{3.43}$$

But, by (2.2) and convexity,

$$\begin{aligned}&\tfrac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, |m(y)|^2 \\ &\qquad \leqslant \tfrac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, m^2_\beta + m_\beta\, \|m - m_\beta s\|^2_{T^c}\end{aligned} \tag{3.44}$$

and analogously

$$\frac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, |m''_{n_k}(y)|^2$$

$$\leqslant \frac{1}{2}\int_T dx \int_{T^c} dy\, J(|x-y|)\, m_\beta^2 + m_\beta\, \|m''_{n_k} - m_\beta s'_{n_k}\|^2_{T^c} \qquad (3.45)$$

Collecting together (3.43), (3.44), (3.45), using (3.40) and the first inequality in (3.41) we finally get $\alpha \geqslant \mathscr{F}(m) - \varepsilon$ and then by the arbitrarity of ε, $\alpha \geqslant \mathscr{F}(m)$. That is $\mathscr{F}$ is lower semicontinuous.

3.3. Upper Bound

Following ref. 12 we prove a crucial estimate for the probability of cylinder sets in $\mathscr{M}$ with basis a finite interval Proposition 3.3 below, from which the upper bound follows by using the properties of $\mathscr{F}$. The basic idea is to perform the block spin transformation in the (random) interval T containing the origin and defined by the condition: the right and the left unit intervals adjacent to T are the first ones where the spin configuration is at "local equilibrium", in a suitable sense to be precised. Then the system in T looks finite and, on the other hand, we can prove that the probability of having local equilibrium not far away from the origin is large enough.

First of all we give a precise notion of local equilibrium. It is reasonable to consider a spin configuration at equilibrium in a given (macroscopic) interval I and with some accuracy $\zeta > 0$ if all the block spins on I are in a ζ-neighbor of an equilibrium magnetization. With respect to the case of Ising systems we have here a continuum of possible equilibrium magnetizations, indexed by the vectors $m_\beta s$, $s \in S^{q-1}$. Therefore it is necessary to discretize the number of possible magnetizations. But we cannot introduce simply a partition of S^{q-1} and define the local equilibrium accordingly. In fact, due again to the continuum of the equilibrium states, the energy cost of some non equilibrium configurations can be very small, independently on the accuracy ζ. Then the probability of finding local equilibrium is not large and the strategy of the proof cannot be applied. To overcome this problem we introduce a weaker definition of equilibrium, obtained by introducing a finer discretization of S^{q-1}, so that the ζ-neighbors of two close magnetizations are overlapping.

For any $\zeta > 0$ we choose $\ell^* \in \mathbb{N}$ and $s_1, ..., s_{\ell^*} \in S^{q-1}$ such that, for any $s \in S^{q-1}$ there is s_ℓ with $|s - s_\ell| \leqslant (1 + m_\beta)^{-1}\, \zeta/4$ (the precise value $(1 + m\beta)^{-1}/4$ comes out for technical reasons, see the proof of Lemma 3.4 below). Notice that we can assume $\ell^* \sim \zeta^{1-q}$. Given δ, $\zeta > 0$ we say that in an interval $[k, k+1) \in \mathscr{D}^{(1)}$ there is equilibrium with accuracy ζ and coarse grain δ if, for any $x \in [k, k+1)$, $|\sigma_\gamma^{(\delta)}(x) - m_\beta s_\ell| < \zeta$ for some s_ℓ. To summarize this we introduce the variable $\eta_{\delta,\zeta}(\cdot,\cdot)$: $\mathbb{Z} \times \mathscr{M} \to \{0, 1, ..., \ell^*\}$ defined as follows. Consider the random subsets of $\{1, ..., \ell^*\}$,

$$L(k, m) = \{\ell: |m^{(\delta)}(x) - m_\beta s_\ell| \leqslant \zeta \text{ for any } x \in [k, k+1)\} \tag{3.46}$$

Then

$$\eta_{\delta,\zeta}(k, m) = \begin{cases} \min L(k, m) & \text{if } \quad L(k, m) \neq \varnothing \\ 0 & \text{otherwise} \end{cases} \tag{3.47}$$

so that $\eta_{\delta,\zeta}(k, \sigma_\gamma) \neq 0$ means that there is equilibrium in $[k, k+1)$ in the sense stated above. When useful we will consider $\eta_{\delta,\zeta}(\cdot,\cdot)$ a function on $\mathbb{Z} \times \mathscr{S}$ by setting, with an abuse of notation, $\eta_{\delta,\zeta}(k, \sigma) = \eta_{\delta,\zeta}(k, \sigma_\gamma)$.

For any pair of unit vectors s, $s' S^{q-1}$ we define the function

$$\chi_{ss'}(x) = \begin{cases} m_\beta s & \text{if } \quad x \geqslant 0 \\ m_\beta s' & \text{if } \quad x < 0 \end{cases} \tag{3.48}$$

and, for any finite interval $I \subset \mathbb{R}$ we introduce the subset of $\mathscr{M}$,

$$\begin{aligned} \mathscr{M}_I = \{m \in \mathscr{M}: \exists \ell_+, \ell_- \in \{1, ..., \ell^*\} \\ \text{such that } m(x) - \chi_{\ell_+\ell_-}(x) = 0 \text{ when } x \in I^c\} \end{aligned} \tag{3.49}$$

where we shorthand $\chi_{s_{\ell_+} s_{\ell_-}} = \chi_{\ell_+\ell_-}$. We also write $\mathscr{M}_r$ for $\mathscr{M}_{[-r,r]}$, $r \in \mathbb{R}$. Then:

Proposition 3.3. For any interval $I = [-a, a]$ with $a \in \mathbb{N}$, any cylinder set F with basis I, any $\zeta > 0$, any δ, $\delta^* \in \{2^{-n}; n \in \mathbb{N}\}$ and any integer $R > a$,

$$\begin{aligned} \limsup_{\gamma \downarrow 0} \gamma \log \mu_{\beta,\gamma}(F) \leqslant -\inf_{m \in F \cap \mathscr{M}_R} \beta \mathscr{F}(m^{(\delta^*)}) + c_1 R \delta^* \\ + c_2(\zeta + \delta) + \vartheta(\zeta, \delta, R - a) \end{aligned} \tag{3.50}$$

where c_1, c_2 and $\vartheta(\zeta, \delta, R)$ are positive and independent on I and F, and $\vartheta(\zeta, \delta, R) \to 0$ as $R \to +\infty$ for any ζ and δ.

Proof. Fix ζ, $\delta > 0$ and $R > a$. Let $\tau_\pm : \mathcal{M} \to \mathbb{N} \cup \{+\infty\}$ be the functions

$$\begin{cases} \tau_+(m) = \inf\{k \in \mathbb{N} : k \geqslant a, \eta_{\delta,\zeta}(k, m) \neq 0\} \\ \tau_-(m) = \inf\{k \in \mathbb{N} : k \geqslant a, \eta_{\delta,\zeta}(-k-1, m) \neq 0\} \end{cases} \tag{3.51}$$

($\tau_\pm(m) = +\infty$ if the infimum is not achieved. For any $k_+, k_- \geqslant a$ and $\ell_+, \ell_- = 1, ..., \ell^*$ we introduce the set

$$G^{(\delta,\zeta)}_{k_\pm,\ell_\pm} = \{m \in m: \tau_\pm = k_\pm, \eta_{\delta,\zeta}(k_+, m) = \ell_+, \eta_{\delta,\zeta}(-k_- 1, m) = \ell_-\} \tag{3.52}$$

and we partition $\mathcal{M}$ as the union of

$$D = \bigcup_{k_+=a}^{R} \bigcup_{k_-=a}^{R} \bigcup_{\ell_+=1}^{\ell^*} \bigcup_{\ell_-=1}^{\ell^*} G^{(\delta,\zeta)}_{k_\pm,\ell_\pm} \tag{3.53}$$

and its complementary set D^c. Recalling (2.7) we introduce the set $\Lambda = \{i \in \mathbb{Z} : |i| \leqslant L\}$, L an integer greater than $\gamma^{-1}R$ that will go to $+\infty$, and we look for an upper bound of $Z^\Lambda_{\beta,\gamma}(F)$, defined as in (3.1) with free boundary conditions. We estimate

$$Z^\Lambda_{\beta,\gamma}(F) \leqslant (R-a)^2 (\ell^*)^2 \max_{\substack{k_\pm = a,..., R \\ \ell_\pm = 1, ..., l^*}} Z^\Lambda_{\beta,\gamma}(F \cap G^{(\delta,\zeta)}_{k_\pm,\ell_\pm}) + Z^\Lambda_{\beta,\gamma}(D^c) \tag{3.54}$$

and we bound first $Z^\Lambda_{\beta,\gamma}(F \cap G^{(\delta,\zeta)}_{k_{pm},\ell_\pm})$ and then $Z^\Lambda_{\beta,\gamma}(D^c)$.

Fix $k_\pm$, $\ell_\pm$ and decompose $\Lambda = \Lambda_+ \cup \Delta \cup \Lambda_-$ with

$$\begin{aligned} &\Delta = \{i \in \mathbb{Z} : -k_- \leqslant i\gamma < k_+\}, \qquad \Lambda_+ = (\Lambda \backslash \Delta) \cap \mathbb{Z}_{>0}, \\ &\Lambda_- = (\Lambda \backslash \Delta) \cap \mathbb{Z}_{<0} \end{aligned} \tag{3.55}$$

Then

$$\begin{aligned} Z^\Lambda_{\beta,\gamma}(F \cap G^{(\delta,\zeta)}_{k_\pm,\ell_\pm}) \leqslant \max_{\sigma \in G^{(\delta,\zeta)}_{k_\pm,\ell_\pm}} & Z^{\Delta,\sigma_{\Delta^c}}_{\beta,\gamma}(F)\, Z^{\Lambda_-}_{\beta,\gamma}(\eta_{\delta,\zeta}(-k_- 1, \cdot) = \ell_-) \\ & \times Z^{\Lambda_+}_{\beta,\gamma}(\eta_{\delta,\zeta}(k_+, \cdot) = \ell_+) \end{aligned} \tag{3.56}$$

We notice that $\sigma \in G^{(\delta,\zeta)}_{k_\pm,\ell_\pm}$ implies $|\sigma^{(\delta^*)}_\gamma(x) - \chi_{\ell_+\ell_-}(x)| \leqslant \zeta$ for any $x \in [-k_- 1, -k_-) \cup [k_+, k_+ + 1)$. Then, calling $T = [-k_-, k_+]$, for some $b_3 > 0$ depending only on J and for any $m \in \mathcal{M}$,

$$
\begin{aligned}
&|U_{T^c}((\sigma_y^{(\delta^*)})_{T^c}) + W_T(m^{(\delta^*)}{}_T \mid (\sigma_y^{(\delta^*)})_{T^c}) \\
&\qquad - U_{T^c}((\chi_{\ell_+\ell_-})_{T^c}) - W_T(m_T^{(\delta^*)} \mid (\chi_{\ell_+\ell_-})_{T^c})| \, \mathbb{1}_{\sigma \in G^{(\delta,\zeta)}_{k_\pm,\ell_\pm}}(\sigma) \\
&\qquad \leqslant b_3(\zeta+\delta) \qquad (3.57)
\end{aligned}
$$

Now we apply Lemma 3.1 to bound $Z^{\Delta,\sigma_{\Delta^c}}_{\beta,\gamma}(F)$. From (3.7) and (3.57) we obtain:

$$
\begin{aligned}
&\max_{\sigma \in G^{(\delta,\zeta)}_{k_\pm,\ell_\pm}} Z^{\Delta,\sigma_{\Delta}^c}_{\beta,\gamma}(F) \\
&\qquad \leqslant \exp[\beta\gamma^{-1}(b_1\delta^* |T| + b_3(\zeta+\delta))](1+4\pi(\delta^*\gamma^{-1})^{b_2})^{|T|/\delta^*} \\
&\qquad \times \exp[-\beta\gamma^{-1}(U_{T^c}((\chi_{\ell_+\ell_-})_{T^c}) + f_\beta(m_\beta)\,|T|)] \\
&\qquad \times \exp[-\beta\gamma^{-1} \inf_{m \in F} \{\mathscr{F}_T(m_T^{(\delta^*)}) + W_T(m_T^{(\delta^*)} \mid (\chi_{\ell_+\ell_-})_{T^c})\}] \qquad (3.58)
\end{aligned}
$$

We are left with the bound on the product of the partition functions in (3.56) coming from the integration over the spin configurations a_{Λ_+} and σ_{Λ_-}. As in ref. 12 we follow Ruelle's strategy,[32,33] and we reconstruct the full partition function on Λ. First of all we use the symmetry of the interaction with respect to global spin rotations to claim that

$$
Z^{\Lambda_-}_{\beta,\gamma}(\eta_{\delta,\zeta}(-k_-1,\cdot)=\ell_-) = Z^{\Lambda_-}_{\beta,\zeta}(\eta_{\delta,\zeta}(-k_-1,\cdot)=\ell_+) \qquad (3.59)
$$

Let now $V_{\delta^*,\rho,T}(m_\beta s_{\ell_+})$ be as in (3.8). From the explicit form of the functional, see also the proof of Lemma 3.4, one easily checks that, for some constant $b_4 > 0$,

$$
0 \leqslant \sup_{m \in V_{\delta^*,\rho,T}(m_\beta s_{\ell_+})} \{\mathscr{F}_T(m_T^{(\delta^*)}) + W_T(m_T^{(\delta^*)} \mid (m_\beta s_{\ell_+})_{T^c})\} \leqslant b_4\rho^2\,|T| \qquad (3.60)
$$

We choose ρ so small that $b_4\rho^2 \leqslant b_1\delta^*$. Then from (3.9), (3.57) and (3.60), for any σ such that $\eta_{\delta,\zeta}(-k_-1,\sigma) = \eta_{\delta,\zeta}(k_+,\sigma) = \ell_+$, we get

$$
\begin{aligned}
Z^{\Delta,\sigma_{\Lambda\setminus\Delta}}_{\beta,\gamma} &\geqslant \exp[-\beta\gamma^{-1}(2b_1\,\delta^*\,|T| + b_3(\zeta+d))]\left(\frac{4\pi\rho^3}{3}(\delta^*\gamma^{-1})^{-c(m_\beta+\rho)}\right)^{|T|/\delta^*} \\
&\qquad \times \exp[-\beta\gamma^{-1}(U_{T^c}((m\beta s_{\ell_+})_{T^c}) + f_\beta(m_\beta)\,|T|)] \qquad (3.61)
\end{aligned}
$$

Now we come back to the r.h.s. of (3.56). We first use (3.59) and then (3.61) to reinsert the integration over the σ_Δ's in order to reconstruct

$Z^{\Lambda}_{\beta,\gamma}(\eta_{\delta,\zeta}(-k_-1,\cdot)=\eta_{\delta,\zeta}(k_+,\cdot)=\ell_+)$ that can be bounded with the full partition function. From (3.58), recalling the definitions (3.4), (3.5), (3.6), (3.49) and that $|T|\leqslant 2R$, we finally obtain

$$Z^{\Lambda}_{\beta,\gamma}(F\cap G^{(\delta,\zeta)}_{k_\pm,\ell_\pm})\leqslant \exp[\beta\gamma^{-1}(6b_1\,\delta^*R+2b_3(\zeta+\delta))]$$
$$\times\left(\frac{3}{4\pi\rho^3}(\delta^*\gamma^{-1})^{c(m_\beta+\rho)}\,(1+4\pi(\delta^*\gamma^{-1})^{\frac{b}{2}}_{2})\right)^{-2R/\delta^*}$$
$$\times\exp[-\beta\gamma^{-1}\inf_{m\in F\cap\mathscr{m}_R}\{\mathscr{F}(m^{(\delta^*)})\}]\,Z^{\Lambda}_{\beta,\gamma} \tag{3.62}$$

To bound $Z^{\Lambda}_{\beta,\gamma}(D^c)$ we need a lower bound for the cost in free energy of a run of non equilibrium configurations which is the content of the following lemma.

Lemma 3.4. Let $T=[t_1,t_2]$, $t_1,t_2\in\mathbb{Z}$. For any $\delta\in\{2^{-n};n\in\mathbb{N}\}$, $\zeta\in(0,m_\beta)$, let $\eta_{\delta,\zeta}(\cdot,\cdot)$ be as in (3.47) and define the cylinder set

$$\Gamma^{(\delta,\zeta)}=\{m\in\mathscr{M}:\eta_{\delta,\zeta}(k,m)=0\text{ for any }k=t_1,\ldots,t_2-1\} \tag{3.63}$$

Then there are positive functions $p(\delta,\zeta)$ and $p^*(\delta,\zeta)$ such that, for any $\delta^*\in\{2^{-n};n\in\mathbb{N}\}$, $\delta^*\leqslant p^*(\delta,\zeta)$,

$$\inf_{m\in\Gamma^{(\delta,\zeta)}}\mathscr{F}_T(m_T^{(\delta^*)})\geqslant p(\delta,\zeta)\,|T| \tag{3.64}$$

where $\mathscr{F}_T$ is defined in (3.4).

We postpone the proof of the above lemma at the end of the subsection and we complete the proof of Proposition 3.3. As before we take $T=[-k_-,k_+]$ with now $k_\pm=\min\{R;\tau_\pm\}$. Since there is not equilibrium at the border of T, an estimate as (3.57) is not still true. We then use that the interactions W_T and U_{T^c} can be uniformly bounded by some constant $C>0$ and we simply estimate

$$Z^{\Lambda}_{\beta,\gamma}(D^c)\leqslant\exp[\beta\gamma^{-1}(6b_1\,\delta^*R+2C)]$$
$$\times\left(\frac{3}{4\pi\rho^3}(\delta^*\gamma^{-1})^{c(m_\beta+\rho)}\,(1+4\pi(\delta^*\gamma^{-1})^{\frac{b}{2}}_{2})\right)^{2R/\delta^*}$$
$$\times\exp[-\beta\gamma^{-1}\inf_{m\in D^c}\{\mathscr{F}_T(m_T^{(\delta^*)})\}]\,Z^{\Lambda}_{\beta,\gamma} \tag{3.65}$$

But, from the definition (3.53), $\min\{k_-; k_+\} = R$, therefore D^c is contained into a set $\Gamma^{(\delta,\zeta)}$ as defined in (3.63) with $t_2 - t_1 \geqslant (R-a)$. From (3.54), (3.62), (3.65) and using Lemma 3.4 we finally get, uniformly in Λ,

$$\begin{aligned}\mu^{\Lambda}_{\beta,\gamma}(F) \leqslant\ & (R-a)^2 (\ell^*)^2 \exp[\beta\gamma^{-1}(6b_1\delta^* R + 2b_3(\zeta+\delta))] \\ & \times \left(\frac{3}{4\pi\rho^3}(\delta^*\gamma^{-1})^{c(m_\beta+\rho)}(1+4\pi(\delta^*\gamma^{-1})^{\frac{b}{2}})\right)^{2R/\delta^*} \\ & \times \max\{\exp[-\beta\gamma^{-1} \inf_{m\in F\cap \mathcal{m}_R} \{\mathcal{F}(m^{(\delta^*)})\}]; \\ & \qquad \exp[-\beta\gamma^{-1}(p(\delta,\zeta)(R-a)-2C)]\} \end{aligned} \tag{3.66}$$

from which the Proposition follows immediately. ∎

With Proposition 3.3 and the lower semicontinuity of the functional $\mathcal{F}$, the proof of (2.32) is exactly the same as in ref. 12, then we omit the proof.

Proof of Lemma 3.4. Let

$$\mathcal{N}_k = \{n\in\mathbb{Z}: k\leqslant n\,\delta < k+1\}, \qquad \mathcal{N} = \bigcap_{k=t_1}^{t_2-1} \mathcal{N}_k$$

From the definition (3.47),

$$\Gamma^{(\delta,\zeta)} = \bigcap_{k=t_1}^{t_2-1} \bigcap_{\ell=1}^{\ell^*} \bigcup_{n\in\mathcal{N}_k} \Gamma^{(\delta,\zeta)}_{n,\ell} \tag{3.67}$$

where, for any $\delta\in\{2^{-n}; n\in\mathbb{N}\}$, $\zeta\in(0,m_\beta)$, $n\in\mathbb{Z}$, $\ell=1,\dots,\ell^*$,

$$\Gamma^{(\delta,\zeta)}_{n,\ell} = \{m\in\mathcal{m} : |m^{(\delta)}(n\,\delta) - m_\beta s_\ell| > \zeta\} \tag{3.68}$$

Now let $\delta^*\in\{2^{-n}; n\in\mathbb{N}\}$, $\delta^*<\delta$, and define, for any $n\in\mathcal{N}$,

$$\mathcal{U}_n = \{u\in\mathbb{Z} : n\delta \leqslant u\delta^* < (n+1)\delta\}, \qquad \mathcal{U} = \bigcup_{n\in\mathcal{N}} \mathcal{U}_n$$

Given $m\in\mathcal{M}$, let $B_{n,\ell}(m) = \{u\in\mathcal{U}_n : |m^{(\delta^*)}(u\,\delta^*) - m_\beta s_\ell| > \zeta/2\}$. Clearly, for any $m\in\Gamma^{(\delta,\zeta)}_{n,\ell}$,

$$\begin{aligned}\zeta < |m^{(\delta)}(n\,\delta) - m_\beta s_\ell| &\leqslant \frac{\delta^*}{\delta}\sum_{u\in\mathcal{U}_n} |m^{(\delta^*)}(u\,\delta^*) - m_\beta s_\ell| \\ &\leqslant 2\frac{\delta^*}{\delta}|B_{n,\ell}(m)| + \frac{\zeta\delta^*}{2\delta}\left(\frac{\delta}{\delta^*} - |B_{n,\ell}(m)|\right)\end{aligned} \tag{3.69}$$

so that $|B_{n,\ell}(m)| > \zeta\delta/(4\delta^*)$. Then

$$\Gamma_{n,\ell}^{(\delta,\zeta)} \subseteq \bigcup_{\substack{U \subseteq \mathscr{U}_n \\ |U| > \zeta\delta/(4\delta^*)}} \bigcap_{u \in U} \Gamma_{u,\ell}^{(\delta^*,\zeta/2)} \tag{3.70}$$

Let $\zeta' = (1+m_\beta)^{-1}\zeta/2$ and $\zeta'' = m_\beta\zeta'/8$. For any $u \in \mathscr{U}$ define

$$A_\ell^{(\delta^*,\zeta')}(u) = \left\{ m \in \mathscr{M} : m^{(\delta^*)}(u\delta^*) \neq 0, \left| \frac{m^{(\delta^*)}(u\delta^*)}{|m^{(\delta^*)}(u\delta^*)|} - s_\ell \right| \leqslant \zeta' \right\}$$

$$B^{(\delta^*,\zeta'')}(u) = \{ m \in \mathscr{M} : \left| |m^{(\delta^*)}(u\delta^*)| - m_\beta \right| \leqslant \zeta'' \}$$

By triangular inequality and since $\zeta \in (0, m_\beta)$ we have

$$A_\ell^{(\delta^*,\zeta')}(u) \cap B^{(\delta^*,\zeta'')}(u) \subseteq (\Gamma^{(\delta^*,\zeta/2)}{}_{u,\ell})^c \tag{3.71}$$

Then, setting

$$R^{(\delta^*,\zeta)}(k) = \bigcup_{n \in \mathscr{N}_k} \bigcup_{\substack{U \subseteq \mathscr{U}_n \\ |U| > \zeta\delta/(4\delta^*)}} \bigcap_{u \in U} B^{(\delta^*,\zeta'')}(u)^c$$

$$\tilde{D}^{(\delta^*,\zeta)}(k) = \bigcap_{\ell=1}^{\ell^*} \bigcup_{n \in \mathscr{N}_k} \bigcup_{\substack{U \subseteq \mathscr{U}_n \\ |U| > \zeta\delta/(4\delta^*)}} \bigcap_{u \in U} A_\ell^{(\delta^*,\zeta')}(u)^c \cap B^{(\delta^*,\zeta'')}(u)$$

from (3.67) and (3.71) we obtain

$$\Gamma^{(\delta,\zeta)} \subseteq \bigcap_{k=t_1}^{t_2-1} R^{(\delta^*,\zeta)}(k) \cup \tilde{D}^{(\delta^*,\zeta)}(k) \tag{3.72}$$

Let now $m \in \tilde{D}^{(\delta^*,\zeta)}(k)$. Then there are $n_1 \in \mathscr{N}_k$ and $U \subseteq \mathscr{U}_{n_1}$, $|U| > \zeta\delta/(4\delta^*)$, such that

$$m \in \bigcap_{u \in U} B^{(\delta^*,\zeta'')}(u) \tag{3.73}$$

In particular the unit vectors $m^{(\delta^*)}(u\,\delta^*)/|m^{(\delta^*)}(u\,\delta^*)| \neq 0$ are well defined for any $u \in U$. But from the definition of $\{s_\ell;\, \ell = 1,\dots,\ell^*\}$, for any $s \in S^{q-1}$, there is s_ℓ such that $|s - s_\ell| < \zeta'/2$. Then we can apply the "pigeon holes principle" to conclude that there must be at least an index ℓ_1 and a set $U_1 \subseteq U$ such that $|U_1| > \zeta\,\delta/(4\ell^*\,\delta^*)$ and

$$m \in \bigcap_{u \in U_1} A_{\ell_1}^{\delta^*,\zeta'/2)}(u) \cap B^{(\delta^*,\zeta'')}(u) \tag{3.74}$$

Since $m \in \tilde{D}^{(\delta^*, \zeta)}(k)$ it follows also that there are $n_2 \in \mathcal{N}_k$ and $U_2 \subseteq \mathcal{U}_{n_2}$, $|U_2| > \zeta\, \delta/(4\delta^*)$ such that

$$m \in \bigcap_{u \in U_2} A_{\zeta_1}^{(\delta^*, \zeta')}(u)^c \cap B^{(\delta^*, \zeta'')}(u) \tag{3.75}$$

From (3.74) and (3.75), for any $u_1 \in U_1$ and $u_2 \in U_2$,

$$\left| \frac{m^{(\delta^*)}(u_1 \delta^*)}{|m^{(\delta^*)}(u_1 \delta^*)|} - \frac{m^{(\delta^*)}(u_2 \delta^*)}{|m^{(\delta^*)}(u_2 \delta^*)|} \right| > \zeta' - \frac{\zeta'}{2} = \frac{\zeta'}{2} \tag{3.76}$$

Since $m \in B^{(\delta^*, \zeta'')}(u)$ for any $u \in U_1 \cup U_2$, see (3.74) and (3.75), from (3.76) and recalling that $\zeta'' - m_\beta \zeta'/8$, we get

$$|m^{(\delta^*)}(u_1 \delta^*) - m^{(\delta^*)}(u_2 \delta^*)| > m_\beta \frac{\zeta'}{2} - 2\zeta'' = 2\zeta''$$

So we found

$$\tilde{D}^{(\delta^*, \zeta)}(k) \subseteq D^{(\delta^*, \zeta)}(k)$$

$$= \bigcup_{n_1, n_2 \in \mathcal{N}_k} \quad \bigcup_{\substack{U_1 \subseteq \mathcal{U}_{n_1} \\ |U_1| > \zeta\delta/(4\ell^*\delta^*)}} \quad \bigcup_{\substack{U_2 \subseteq \mathcal{U}_{n_2} \\ |U_2| > \zeta\delta/(4\delta^*)}} \quad \bigcap_{\substack{u_1 \in U_1 \\ u_2 \in U_2}} C^{(\delta^*, \zeta'')}(u_1, u_2) \tag{3.77}$$

where

$$C^{(\delta^*, \zeta'')}(u_1, u_2) = \{ m \in \mathcal{M} \colon |m^{(\delta^*)}(u_1 \delta^*) - m^{(\delta^*)}(u_2 \delta^*)| > 2\zeta'' \} \tag{3.78}$$

From (3.72) and (3.77) we finally obtain

$$\Gamma^{(\delta^*, \zeta)} \subseteq \bigcap_{k = t_1}^{t_2 - 1} R^{(\delta^*, \zeta)}(k) \cup D^{\delta^*, \zeta)}(k) \tag{3.79}$$

Now it is easy to get a lower bound for $\mathcal{F}_T(m_T^{(\delta^*)})$ on the set in the r.h.s; of (3.79) (and so also in $\Gamma^{(\delta, \zeta)}$. Recalling (3.4),

$$\mathcal{F}_T(m_T) \geqslant \sum_{k = t_1}^{t_2 - 1} (\mathcal{F}_k^{(R)}(m^{(\delta^*)}) + \mathcal{F}_k^{(D)}(m^{(\delta^*)})) \tag{3.80}$$

where

$$\mathcal{F}_k^{(R)}(m^{(\delta^*)}) = \sum_{n \in \mathcal{N}_k} \sum_{u \in \mathcal{U}_n} \delta^* (f_\beta(m^{(\delta^*)}(u\delta^*)) - f_\beta(m_\beta))$$

and

$$\mathscr{F}_k^{(D)}(m^{(\delta^*)}) = \frac{1}{4} \sum_{n_1, n_2 \in \mathscr{N}_k} \sum_{\substack{u_1 \in \mathscr{U}_{n_1} \\ u_2 \in \mathscr{U}_{n_2}}} \delta^* J_{\delta^*}(u_1, u_2) \, |m^{(\delta^*)}(u_1 \delta^*) - m^{(\delta^*)}(u_2 \delta^*)|^2$$

Using the fact that m_β is a quadratic absolute minima of f_β and assuming $J = \mathbb{1}_{[0,1]}$, for a suitable positive constants c'.

$$\inf_{m \in R^{(\delta^*, \zeta)}(k)} \mathscr{F}_k^{(R)}(m^{(\delta^*)}) > c' \delta \zeta^3, \qquad \inf_{m \in D^{(\delta^*, \zeta)}(k)} \mathscr{F}_k^{(D)}(m^{(\delta^*)}) > c' \delta^2 \zeta^{q+3} \tag{3.81}$$

(for the second one we used also that $\ell^* \sim \zeta^{-q+1}$). In the general case, since $J > 0$ in $(0, 1)$, the term $c' \delta^2 \zeta^{q+3}$ is replaced by some function $g_J(\delta, \zeta)$ which is strictly positive for $\delta, \zeta > 0$. The lemma follows immediately. ▮

3. Lower Bound

As in ref. 12 we introduce the basis of neighborhoods $\{V_{\delta, \zeta, R}; \delta > 0, \zeta > 0, R \in \mathbb{N}\}$, $m \in \mathscr{M}$, for the weak L_2-loc topology where $V_{\delta, \zeta, R}(m)$ is defined as in (3.8) when $T = [-R, R]$ and $(\delta^*, \rho) = (\delta, \zeta)$. First we prove a probability estimate, Proposition 3.5 below, for neighborhoods of $m \in \mathscr{M}$ such that $\|m\|_\infty < 1$. Next, by exploiting the properties of the free energy functional $\mathscr{F}$ and of the set $\mathscr{M}^0$, see (2.31), we deduce (2.33).

Proposition 3.5. Let $m \in \mathscr{M}$, $\|m\|_\infty < 1$. Suppose there are $R \in \mathbb{N}$ and two unit vectors $s_+, s_- \in S^{q-1}$ so that $m(x) - \chi_{s_+, s_-}(x) = 0$ if $|x| \geqslant R$, see definition (3.48). Then, for any $\varepsilon > 0$ there are $\zeta > 0$ and $\delta \in \{2^{-n}; n \in \mathbb{N}\}$ such that

$$\liminf_{\gamma \downarrow 0} \gamma \log \mu_{\beta, \gamma}(V_{\delta, \zeta, R}(m)) \geqslant -\beta \mathscr{F}(m) - \varepsilon \tag{3.82}$$

Proof. As in the proof of Proposition 3.3, we use (2.7) and we work in finite volume. Let $L \in \mathbb{N}$ larger than $\gamma^{-1} R$ and $\Lambda = \{i \in \mathbb{Z} : |i| \leqslant L\}$. For any $k \in \mathbb{Z}$ we define the events

$$\begin{aligned} G_\pm^{(\delta, \zeta)}(k) &= \{m \in \mathscr{M} : |m^{(\delta)}(x) - m_\beta s_\pm| < \zeta \ \forall x \in [k, k+1)\} \\ G^{(\delta, \zeta)} &= G_+^{(\delta, \zeta)}(R) \cup G_-^{(\delta, \zeta)}(-R-1) \end{aligned} \tag{3.83}$$

and we introduce the sets Δ, $\Lambda_\pm$ as in (3.55) with here $k_\pm = R$. Then we bound

$$Z^{\Delta}_{\beta,\gamma}(V_{\delta,\zeta,R}(m)) \geqslant Z^{\Delta}_{\beta,\gamma}(V_{\delta,\zeta,R}(m) \cap G^{(\delta,\zeta)})$$
$$\geqslant \min_{\sigma_{\Delta^c} \in G^{\delta,\zeta)}} Z^{\Delta,\sigma_{\Delta^c}}_{\beta,\gamma}(V_{\delta,\zeta,R}(m))\, Z^{\Lambda_+}_{\beta,\gamma}(G^{(\delta,\zeta)}_+(R))$$
$$\times Z^{\Lambda_-}_{\beta,\gamma}(G^{(\delta,\zeta)}_-(-R-1)) \tag{3.84}$$

Now we can apply item (ii) of Lemma 3.1 to $Z^{\Delta,\sigma_{\Delta^c}}_{\beta,\gamma}(V_{\delta,\zeta,R}(m))$ with $T=[-R, R]$ and $(\delta^*, \rho) = (\delta, \zeta)$. Using the smoothness properties of the functions $J(\cdot)$ and $I(\cdot)$ and since $\sigma_{\Delta^c} \in G^{(\delta,\zeta)}$, for some constant $b_5 > 0$.

$$\Big| \sup_{\tilde m \in V_{\delta,\rho,T}(m)} \{\mathscr{F}_T(\tilde m^{(\delta)}_T) + W_T(\tilde m^{(\delta)}_T \mid (\sigma^{(\delta)}_\gamma)_{T^c})\} + U_{T^c}((\sigma^{(\delta)}_\gamma)_{T^c})$$
$$- \mathscr{F}_T(m^{(\delta)}_T) - W_T(m^{(\delta)}_T \mid m_{T^c}) - U_{T^c}(m_{T^c})\Big| \leqslant b_5(\zeta+\delta)\,|T| \tag{3.85}$$

(we used that $m^{(\delta)}_{T^c} = m_{T^c} = (\chi_{s_+s_-})_{T^c}$). We need to replace the coarse grain $m^{(\delta)}_T$ with m_T. This can be done in the energy terms $E(\cdot\mid\cdot)$ and $W_T(\cdot\mid\cdot)$ with an error of order δ. Conversely, we have not such an estimate for the (local) entropy term in $\mathscr{F}_T$ because the difference $m(x) - m^{(\delta)}(x)$ is not small in general. But, since the entropy is a convex function,

$$\delta I(m^{(\delta)}(x)) \geqslant \int_{n\delta}^{(n+1)\delta} dy\, I(m(y)) \qquad \forall x \in [n\,\delta, (n+1)\delta) \tag{3.86}$$

From (3.85), (3.86) and item ii) of Lemma 3.1, there is a constant $b_6 > 0$ such that, for any ζ small enough,

$$\min_{\sigma_{\Delta^c} \in G^{(\delta,\zeta)}} Z^{\Delta,\sigma_{\Delta^c}}_{\beta,\gamma}(V_{\delta,\zeta,R}(m)) \geqslant \left(\frac{4\pi\zeta^3}{3}(\delta\gamma^{-1})^{-c(\|m_T\|_\infty+\zeta)}\right)^{2R/\delta}$$
$$\times \exp[-\beta b_6(\delta+\zeta)\,\gamma^{-1}R]$$
$$\times \exp[-\beta\gamma^{-1}(\mathscr{F}(m) + 2Rf_\beta(m_\beta))] \tag{3.87}$$

(we used that $\mathscr{F}_{T^c}(m_{T^c}) = 0$).

Next, analogously to what done when proving Proposition 3.3, we reconstruct the full partition function to bound the other terms in the r.h.s. of (3.84). First we use the rational symmetry to replace $Z^{\Lambda_-}_{\beta,\gamma}(G^{(\delta,\zeta)}_-(-R-1))$ by $Z^{\Lambda_-}_{\beta,\gamma}(G^{(\delta,\zeta)}_+(-R-1))$. Then we use item (i) of Lemma 3.1 with $\Gamma = V_{\delta,\zeta,R}(m_\beta s_{\ell_+})$ and $\sigma_{\Delta^c} \in G^{(\delta,\zeta)}_+(R) \cap G^{(\delta,\zeta)}_+(-R-1)$ to reintroduce the sum over the σ_Δ's. After some simple estimates we get, for some constant $b_7 > 0$.

$$\begin{aligned} &Z^{\Delta_+}_{\beta,\gamma}(G^{(\delta,\zeta)}_+(R))\, Z^{\Delta_-}_{\beta,\gamma}(G^{(\delta,\zeta)}_-(-R-1)) \\ &\quad \geqslant (1+4\pi(\delta\gamma^{-1})^{b_2})^{-2R/\delta} \exp[-\beta b_7(\delta+\zeta)\,\gamma^{-1}R] \\ &\quad \times \exp[\beta\gamma^{-1}2Rf_\beta(m_\beta)]\, Z^{\Delta}_{\beta,\gamma}\left(\bigcap_{k=-R-1}^{R} G^{(\delta,\zeta)}_+(k)\right) \end{aligned} \tag{3.88}$$

From (3.87) and (3.88), taking first the limit $L \to +\infty$ and after $\gamma \downarrow 0$,

$$\begin{aligned} &\liminf_{\gamma\downarrow 0} \gamma \log \mu_{\beta,\gamma}(V_{\delta,\zeta,R}(m)) \\ &\quad \geqslant -\beta\mathscr{F}(m) - \beta(b_6+b_7)(\delta+\zeta)\,R + \liminf_{\gamma\downarrow 0} \gamma \log \mu_{\beta,\gamma}\left(\bigcap_{k=-R-1}^{R} G^{(\delta,\zeta)}_+(k)\right) \end{aligned} \tag{3.89}$$

We are going to prove that the liminf in the r.h.s. of (3.89) is 0, so the Proposition is proved. For any $s \in S^{q-1}$ let

$$G_s = \{m \in \mathscr{m}\colon |m^{(\delta)}(x) - m_\beta s| < \zeta \ \forall x \in [-R-1, R)\} \tag{3.90}$$

Clearly G_{s_+} is the set in the r.h.s. of (3.89) and, by symmetry, $\mu_{\beta,\gamma}(G_s)$ does not depend on s. Fix $n = n(\zeta)$ unit vectors uniformly distributed on S^{q-1} such that the balls $B_{s_i} = \{v \in \mathbb{R}^q\colon |v - m_\beta s_i| < \zeta\}$ are disjoint and $\operatorname{dist}(m_\beta s, \bigcup B_{s_i}) < \zeta$ for any $s \in S^{q-1}$. Then,

$$n\mu_{\beta,\gamma}(G_s) = \mu_{\beta,\gamma}\left(\bigcup_{i=1}^{n} G_{s_i}\right) = 1 - \mu_{\beta,\gamma}\left(\bigcap_{i=1}^{n} G^c_{s_i}\right) \tag{3.91}$$

With an appropriate rotation of the s_i's we can find n new vectors s_i' such that, for any $s \in S^{q-1}$, $\operatorname{dist}(m_\beta s, \bigcap_i B^c_{s_i} \cap B^c_{s_i'}) > c\zeta$ for a suitable constant $c > 0$ depending only on q and β. Since

$$\mu_{\beta,\gamma}\left(\bigcap_{i=1}^{n} G^c_{s_i'}\right) \leqslant n\mu_{\beta,\gamma}(G_s) + \mu_{\beta,\gamma}\left(\bigcap_{i=1}^{n} G^c_{s_i} \cap G^c_{s_i'}\right)$$

from (3.91) we get

$$\mu_{\beta,\gamma}(G_s) \geqslant \frac{1}{2n} - \frac{1}{2n}\mu_{\beta,\gamma}\left(\bigcap_{i=1}^{n} G^c_{s_i} \cap G^c_{s_i'}\right) \tag{3.92}$$

Recalling the definition (3.90), from the construction of the vectors s_i and s_i', one checks that there is $c' > 0$ such that the set in the r.h.s. of (3.92) is contained in the union of the events: $\{\exists x \in [-R-1, R): \mid |m^{(\delta)}(x)| - m_\beta| > c'\zeta\}$ and $\{\exists x, y \in [-R-1, R): |m^{(\delta)}(x) - m^{(\delta)}(y)| > c'\zeta\}$. The latter one is clearly contained in the bigger one: $\{\exists x \in [-R-1, R): |m^{(\delta)}(x) - m^{(\delta)}(x+\delta)| > c'\delta\zeta/(2R)\}$. We can use then the same arguments that lead to Lemma 3.4 to prove that the infimum of $\mathscr{F}_{[-R,R]}(m^{(\delta^*)})$ over the above union of events is bounded from below by $c(\delta, \delta\zeta/R)$ for some $c(\cdot,\cdot) > 0$ and any $\delta^* \in \{2^{-n}; n \in \mathbb{N}\}$. Then, from Proposition 3.3 (with parameters (δ, ζ) smaller than the above one's!), we obtain $\liminf \gamma \log \mu_{\beta,\gamma}(G_s) = 0$. ∎

To use the above result to prove (2.33) we need a preliminary lemma.

Lemma 3.6. Let $m \in \mathscr{M}^0$, so that there exists a map $x \mapsto s(x) \in S^{q-1}$ s.t. $\|m - m_\beta s\| < +\infty$ and $U(s) < +\infty$. Then there is $N_0 \in \mathbb{N}$ such that, for any $z \in \mathbb{Z}$, $|z| \geqslant N_0$, the unit vector

$$s_z \equiv \left| \int_z^{z+1} dx\, s(x) \right|^{-1} \int_z^{z+1} dx\, s(x) \tag{3.93}$$

is well defined and

$$\lim_{|z| \to +\infty} \|m - m_\beta s_z\|_{[z,z+1]} = 0 \tag{3.94}$$

(recall definition (3.24)).

Proof. We first observe that, since $|s(x)| = 1$ and J is positive and satisfies the normalization condition (2.2),

$$1 - \left| \int_z^{z+1} dx\, s(x) \right| \leqslant \int_z^{z+1} dx (1 - s(x) \cdot J * s(x)) \tag{3.95}$$

and the integral in the r.h.s. of (3.95) goes to 0 as $|z| \to +\infty$ since $U(s) < +\infty$. Then s_z is well defined for $|z|$ large. Now we bound

$$\|m - m_\beta s_z\|^2_{[z,z+1]} \leqslant 2\, \|m - m_\beta s\|^2_{[z,z+1]} + 2m_\beta^2\, \|s - s_z\|^2_{[z,z+1]} \tag{3.96}$$

The first norm in the r.h.s. of (3.96) goes to 0 as $|z| \to +\infty$ since $\|m - m_\beta s\| < +\infty$, while

$$\|s - s_z\|^2_{[z,z+1]} = 2\left(1 - \left| \int_z^{z+1} dx\, s(x) \right| \right)$$

goes to 0 as $|z| \to +\infty$ because of (3.95). ∎

Now we prove (2.33). Let A be an open set in $\mathscr{M}$. If $A \cap \mathscr{M}^0 = \emptyset$, (2.33) holds trivially, therefore we assume $A \cap \mathscr{M}^0 \neq \emptyset$. Let $m \in A \cap \mathscr{M}^0$, then there is $x \mapsto s(x) \in S^{q-1}$ such that $\|m - m_\beta s\| < +\infty$ and $U(s) < +\infty$. Let $\rho \in (0, 1 - m_\beta)$, $N \in \mathbb{N}$, $N \geqslant N_0$ and $s_N, s_{-N-1} \in S^{q-1}$ ($N_0, zz \mapsto s_z$ as in Lemma 3.6). We define the set

$$X_\rho = \{x \in \mathbb{R} : |m(x)| > 1 - \rho\}$$

Clearly $|X_\rho| < +\infty$ and $X_\rho \subseteq X_{\rho'}$ if $\rho \leqslant \rho'$. Moreover, recalling the proof of Lemma 3.2, we can assume $s(x) = |m(x)|^{-1} m(x)$ for any $x \in X_\rho$. Setting $T_N = [-N, N]$ we define

$$m_{\rho, N}(x) \equiv \begin{cases} m(x) & \text{if} \quad x \in T_N \backslash X_\rho \\ (1-\rho)\, s(x) & \text{if} \quad x \in T_N \cap X_\rho \\ m_\beta s_N & \text{if} \quad x > N \\ m_\beta s_{-N-1} & \text{if} \quad x < -N \end{cases} \tag{3.97}$$

By the dominated convergence theorem, $m_{\rho, N}$ converges to m weakly in L_2-loc as $\rho \downarrow 0$ and $N \to +\infty$. Then, since A is an open set, $m_{\rho, N} \in A$ for any ρ sufficiently small and N large enough. Recalling the definitions (3.4) and (3.5) one gets

$$\begin{aligned} \mathscr{F}(m) - \mathscr{F}(m_{\rho, N}) = {} & \mathscr{F}_{T_N}(m_{T_N}) - \mathscr{F}_{T_N}((m_{\rho, N})_{T_N}) + \mathscr{F}_{T_N^c}(m_{T_N^c}) \\ & + W_{T_N}(m_{T_N} \mid m_{T_N^c}) - W_{T_N}((m_{\rho, N})_{T_N} \mid (m_{\rho, N})_{T_N^c}) \end{aligned}$$

Since $f_\beta(m)$, as a function of $|m|$, is left continuous in $|m| = 1$, using the definition of $m_{\rho, N}$ in T_N, one easily checks that the difference $\mathscr{F}_{T_N}(m_{T_N}) - \mathscr{F}_{T_N}((m_{\rho, N})_{T_N}) \to 0$ as $\rho \downarrow 0$ uniformly in N. On the other hand, using (2.2) and the support properties of J, after some simple computation one easily gets,

$$\begin{aligned} & |W_{T_N}(m_{T_N} \mid m_{T_N^c}) - W_{T_N}((m_{\rho, N})_{T_N} \mid (m_{\rho, N})_{T_N^c})| \\ & \quad \leqslant \tfrac{1}{2} \int_{T_N} dx \mid |m(x)|^2 - |m_{\rho, N}(x)|^2| + \tfrac{1}{2} \int_{T_{N+1} \backslash T_N} dy \mid |m(y)|^2 - m_\beta^2| \\ & \qquad + \int_{N-1}^{N} dx \int_N^{N+1} dy\, J(|x-y|)\, |m(x) \cdot m(y) - m_\beta m_{\rho, N}(x) \cdot s_N| \\ & \qquad + \int_{-N}^{-N+1} dx \int_{-N-1}^{-N} dy\, J(|x-y|)\, |m(x) \cdot m(y) - m_\beta m_{\rho, N}(x) \cdots {}_{-N-1}| \end{aligned} \tag{3.99}$$

The first integral in the last line of (3.99) is bounded by $\rho \, |X_\rho| \to 0$ as $\rho \downarrow 0$. The second one is bounded by $\|m - m_\beta s\|_{T_N^c} \to 0$ as $N \to +\infty$. The last two integrals can be estimate in a same way, let us consider the first one. By adding and subtracting $m_{\rho, N}(x) \cdot m(y)$ one checks that it is bounded by

$$\int_{N-1}^{N} dx \, |m(x) - m_{\rho, N}(x)| + \|m - m_\beta s_N\|^2_{[N, N+1]}$$

that goes to 0 as $\rho \downarrow 0$ and $N \to +\infty$ (recall (3.94)). We conclude that for any $\varepsilon > 0$ we can choose ρ_ε and N_ε such that $m_{\rho_\varepsilon}, N_\varepsilon \in A$ and

$$\mathscr{F}(m) \geqslant \mathscr{F}(m_{\rho_\varepsilon}, N_\varepsilon) - \frac{\varepsilon}{2} \tag{3.100}$$

Since $m_{\rho_\varepsilon}, N_\varepsilon \in A$ there are δ_ε, ζ_ε and R_ε such that $V_{\delta_\varepsilon}, \zeta_\varepsilon, R_\varepsilon(m_{\rho_\varepsilon}, N_\varepsilon \subseteq A$. Since we can always assume $R_\varepsilon \geqslant N_\varepsilon$, by applying Proposition 3.5, for any δ, ζ small enough,

$$\liminf_{\gamma \downarrow 0} \gamma \log \mu_{\beta, \gamma}(V_{\delta, \zeta, R_\varepsilon}(m_{\rho_\varepsilon}, N_\varepsilon)) \geqslant \mathscr{F}(m_{\rho_\varepsilon, N_\varepsilon}) - \frac{\varepsilon}{2} \tag{3.101}$$

Choosing $\delta \leqslant \delta_\varepsilon$ and $\zeta \leqslant \zeta_\varepsilon$, from (3.100) and (3.101) we find

$$\liminf_{\gamma \downarrow 0} \gamma \log \mu_{\beta, \gamma}(A) \geqslant \mathscr{F}(m) - \varepsilon$$

from which (2.33) follows by the arbitrarily of ε.

4. TYPICAL BEHAVIOR

In this section we prove Theorems 2.3, 2.4 and 2.5

Proof of Theorem 2.3. Since $\mu_{\beta, \gamma}$ is shift invariant, recalling the definitions (2.22) and (3.47), we can bound

$$\begin{aligned} &\mu_{\beta, \gamma}(\{ \, | \, |m^{(\delta)}(x)| - m_\beta| \leqslant \zeta \text{ for any } |x| \leqslant \gamma^{-p} \}) \\ &\quad \geqslant 1 - 2\gamma^{-1-p} \mu_{\beta, \gamma}(\eta_{\delta, \zeta}(0, \cdot) = 0) \end{aligned} \tag{4.1}$$

Using Proposition 3.3 (with parameters (δ, ζ) much smaller than the above one's) and Lemma 3.4, there is $c > 0$, depending on δ and ζ, such that $\mu_{\beta, \gamma}(\eta_{\delta, \zeta}(0, \cdot) = 0) \leqslant \exp[-c\gamma^{-1}]$ for any γ small enough. Therefore (2.24) follows from (4.1).

Now we prove (2.25). By standard density arguments it is sufficient to prove the convergence of the distribution of the spins in a finite interval $\Delta \subset \mathbb{Z}$ that we can assume centered in the origin. By the DLR equations, the Radon–Nikodym derivative of $\mu_{\beta,\gamma}(d\sigma_\Delta)$ with respect to the a priori product measure on $\mathscr{S}_\Delta$ is

$$\mathscr{Q}_{\beta,\gamma}(\sigma_\Delta) = \int \mu_{\beta,\gamma}(d\sigma_{\Delta^c}) \frac{1}{Z_\beta^{\Delta,\sigma_{\Delta^c}},\gamma} \times \exp\left[-\beta H_\gamma(\sigma_\Delta) + \beta \sum_{i\in\Delta} h_\gamma(i \mid \sigma_{\Delta^c}) \cdot \sigma_i \right] \tag{4.2}$$

where $h_\gamma(i \mid \sigma_{\Delta^c}) \equiv \sum_{j\in\Delta^c} J_\gamma(i-j)\,\sigma_j$. Since Δ is fixed independent on γ and $|\sigma_k| = 1$ for any $k \in \mathbb{Z}$,

$$\lim_{\gamma\downarrow 0} \sup_{\sigma_\Delta} H_\gamma(\sigma_\Delta) = 0, \qquad \lim_{\gamma\downarrow 0} \max_{i\in\Delta} \sup_{\sigma_{\Delta^c}} |h_\gamma(i \mid \sigma_{\Delta^c}) - h_\gamma(0 \mid \sigma^{\Delta^c})| = 0$$

so that

$$\mathscr{Q}_{\beta,0^+}(\sigma_\Delta) = \lim_{\gamma\downarrow 0} \int \mu_{\beta,\gamma}(d\sigma_{\Delta^c}) \prod_{\in\Delta} \phi(\beta h_\gamma(\sigma_{\Delta^c}))^{-1} \exp[\beta h_\gamma(\sigma_{\Delta^c}) \cdot \sigma] \tag{4.3}$$

where we shorthand $h_\gamma(0 \mid \sigma_{\Delta^c}) = h_\gamma(\sigma_{\Delta^c})$ and $\phi(\cdot)$ is defined in (2.12). By using the symmetry of the Gibbs measure with respect to global rotations, (4.3) gives

$$\mathscr{Q}_{\beta,0^+}(\sigma_\Delta) = \lim_{\gamma\downarrow 0} \int \nu(ds) \int \mu_{\beta,\gamma}(d\sigma_{\Delta^c}) \prod_{i\in\Delta} \phi(\beta\, |h_\gamma(\sigma_{\Delta^c})| s)^{-1} \times \exp[\beta\, |h_\gamma(\sigma_{\Delta^c})|\, s \cdot \sigma_i] \tag{4.4}$$

But from (2.2) and (4.1), for any $\zeta > 0$,

$$\lim_{\gamma\downarrow 0} \mu_{\beta,\gamma}(\{\, ||h_\gamma(\sigma_{\Delta^c})| - m_\beta| < \zeta \}) = 1 \tag{4.5}$$

By (4.4), (4.5) and the arbitrarity of ζ we finally get

$$\mathscr{Q}_{\beta,0^+}(\sigma_\Delta) = \int \nu(ds) \prod_{i\in\Delta} \phi(\beta m_\beta s)^{-1} \exp[\beta m_\beta s \cdot \sigma_i] \tag{4.6}$$

that proves (2.25) since. from the mean field equation (2.18), $\beta m_\beta s = t^*(m_\beta)\, s = h^*(m_\beta s)$. ∎

Proof of the Theorem 2.4. Using the translation invariance of the system, we have

$$\mu_{\beta,\gamma}(\{ | \, |m^{(\delta'')}(x)| - m_\beta| \geqslant \zeta'' \text{ for some } |x| \leqslant e^{c\gamma^{-1+\alpha(1-\varepsilon)}}\})$$
$$\leqslant 2(\delta'')^{-1} e^{c\gamma^{-1+\alpha(1-\varepsilon)}} \mu_{\beta,\gamma}(\{ | \, |m^{(\delta'')}(0)| - m_\beta| \geqslant \zeta''\}) \tag{4.7}$$

To estimate this last probability, we can use the estimate (3.66) with parameters $\tilde{\delta}''$ and $\tilde{\zeta}''$ much smaller than the parameters δ'' and ζ''. Using the first inequality in (3.81) to bound the infimum of the rate functional, after some easy estimates we get that, for some constant $\bar{c} > 0$,

$$\mu_{\beta,\gamma}(\{ | \, |m^{(\delta'')}(0) \, | - m_\beta| \geqslant \zeta''\})$$
$$\leqslant (R'')^2 \ell^*(\tilde{\zeta}'')^2 \exp[-\beta\gamma^{-1}(c'\delta''(\zeta'')^3 - \bar{c}(\tilde{\delta}'' + \tilde{\zeta}'')$$
$$- \bar{c}R''(\delta^* + \gamma(\delta^*)^{-1} \log(\delta^*\gamma^{-1})))]$$
$$+ \exp[-\beta\gamma^{-1}(c'R''(\tilde{\delta}'')^2 (\tilde{\zeta}'')^{3+q} - 2C$$
$$- \bar{c}R''(\delta + \gamma(\delta^*)^{-1} \log(\delta^*\gamma^{-1})))] \tag{4.8}$$

Note that we have taken the worst term for the estimates of $p(\tilde{\delta}'', \tilde{\zeta}'')$, see (3.81). The next step is just to make an appropriate choice of all the parameters. Let us first remark that in order that the last exponential goes to zero, we need

$$R'' \geqslant \frac{c_1}{(\tilde{\delta}'')^2 (\tilde{\zeta}'')^{3+q}} \tag{4.9}$$

for some constant $c_1 > 2C/c'$, and also, to compensate the entropy term,

$$c'(\tilde{\delta}'')^2 (\tilde{\zeta}'')^{3+q} \geqslant c_2(\delta^* + \gamma(\delta^*)^{-1} \log(\delta^*\gamma^{-1})) \approx \gamma^{1/2} \log \gamma^{-1/2} \tag{4.10}$$

for some positive constant $c_2 > \bar{c}$. In the last step we have used that the minimum of the term in the middle is reached by choosing $\delta^* \approx (\gamma^{1/2} \log \gamma^{-1/2})$. Looking at the second exponential in the formula (4.8), we want to take the smallest possible $\tilde{\delta}'' + \tilde{\zeta}''$. Given that (4.10) has to be satisfied, it is a simple minimization problem with a constraint, the solution is $\tilde{\delta}'' = c_3\tilde{\zeta}''$ for some positive constant c_3 and this implies that we have to satisfy, for some $c_4 > 0$,

$$\tilde{\zeta}'' > c_4(\gamma^{1/2} \log \gamma^{-1/2})^{(5+q)^{-1}} \tag{4.11}$$

We choose $\tilde{\zeta}'' = \gamma^{d''}$ with $d'' < (2(5+q))^{-1}$. In which case (4.9) is satisfied with $R'' = c_1\gamma^{-(5+q)d''}/c_3^2$.

Now, in order that the first exponential in (4.8) goes to zero, we need at least for suitable $c_5, c_6 > 0$,

$$\delta''(\zeta'')^3 > c_5 \gamma^{(1/2-(5+q)\,d'')} \log \gamma^{-1/2} + c_6 \gamma^{d''} \tag{4.12}$$

Taking $d'' < \frac{1}{2} - (5+q)\, d''$ that is $d'' < (2(6+q))^{-1}$ the condition we get is just

$$\delta''(\zeta'')^3 > (c_5 + c_6)\, \gamma^{d''} \tag{4.13}$$

Clearly the Theorem 2.4 follows with any $c > c_5 + c_6$. ∎

Proof of the Theorem 2.5. It is sufficient to prove that

$$\lim_{\gamma \downarrow 0} \mu_{\beta,\gamma}(\mathscr{R}_L(\Theta, \rho)^c) = 0 \tag{4.14}$$

Using the formula (3.66), for some $c > 0$,

$$\begin{aligned}
&\mu_{\beta,\gamma}(\mathscr{R}_L(\Theta, \rho)^c) \\
&\quad \leqslant R^2 \ell^*(\zeta)^2 \exp[-\beta\gamma^{-1}[\inf_{\mathscr{R}_L(\Theta,\rho)^c \cap \mathscr{M}_R} \mathscr{F}(m^{(\delta^*)}) \\
&\qquad - c(L+R)(\delta^* + \gamma(\delta^*)^{-1} \log(\delta^*\gamma^{-1})) - c(\zeta + \delta)]] \\
&\qquad + \exp[-\beta\gamma^{-1}[cR(c'\zeta^{3+q}\delta^2 - (\delta^* + \gamma(\delta^*)^{-1} \log(\delta^*\gamma^{-1}))) - 2C]]
\end{aligned} \tag{4.15}$$

The next step is to estimate the previous infimum. Note that we have $\mathscr{F}(m^{(\delta^*)}) \geqslant \mathscr{F}_{[-L,+L]}(m^{(\delta^*)})$, moreover

$$\mathscr{F}_{[-L,+L]}(m^{(\delta^*)}) \geqslant U_L(m^{(\delta^*)}) \tag{4.16}$$

where

$$U_L(m) \equiv \frac{1}{4} \int_{-L}^{L} dx \int_{-L}^{L} dy\, J(x-y)\, |m(x) - m(y)|^2 \tag{4.17}$$

Therefore

$$\inf_{m \in \mathscr{R}_L(\Theta,\rho)^c \cap \mathscr{M}_R} \mathscr{F}(m^{(\delta^*)}) \geqslant \inf_{m \in \mathscr{R}_L(\Theta,\rho)^c} U_L(m^{(\delta^*)}) \tag{4.18}$$

Now we have

$$\mathscr{R}_L(\Theta, \rho)^c \subset \Big\{m \in \mathscr{M} : \exists x \in [-L, +L), \\ |m^{(\rho)}(x) - m^{(\rho)} - (x+\rho)| > \frac{\Theta\rho}{2L}\Big\} \equiv \tilde{\mathscr{R}}_L(\Theta, \rho)^c \quad (4.19)$$

For any $x \in [-L, +L)$ let $n = n(x) \in \mathbb{Z}$ be such that $n\rho \leqslant x < (1+1)\,\rho$,

$$\mathscr{U}_x \equiv \{u \in \mathbb{Z} : n \leqslant u\delta^* < (n+1)\,\delta^*\} \quad (4.20)$$

and

$$\mathscr{V}_x \equiv \mathscr{U}_{x+\rho/\delta^*} \quad (4.21)$$

By triangular inequality,

$$|m^{(\rho)}(x) - m^{(\rho)}(x+\rho)| \leqslant \left(\frac{\delta^*}{\rho}\right)^2 \sum_{\substack{u \in \mathscr{U}_x \\ v \in \mathscr{V}_x}} |m^{(\delta^*)}(u\delta^*) - m^{(\delta^*)}(v\delta^*)| \quad (4.22)$$

and, by similar arguments as in (3.69), if we denote by

$$B_x(m) \equiv \left\{(u, v) \in \mathscr{U}_x \times \mathscr{V}_x : |m^{(\delta^*)}(u\delta^*) - m^{(\delta^*)}(v\delta^*)| > \frac{\Theta\rho}{4L}\right\} \quad (4.23)$$

then, for any $m \in \tilde{\mathscr{R}}_L(\Theta, \rho)^c$, we have

$$|B_x(m)| \geqslant \frac{\Theta\rho}{8L}\left(\frac{\rho}{\delta^*}\right)^2 \quad (4.24)$$

from which we get

$$\inf_{m \in \mathscr{R}_L(\Theta, \rho)^c} U_L(m^{(\delta^*)}) \geqslant (\delta^*)^2 \left(\frac{\Theta\rho}{4L}\right)^2 \frac{\Theta\rho}{8L}\left(\frac{\rho}{\delta^*}\right)^2 \equiv \frac{c(\Theta, \rho)}{L^3} \quad (4.25)$$

We insert (4.25) in (4.15) and we find

$$\begin{aligned}
&\mu_{\beta,\gamma}(\mathscr{R}_L(\Theta, \rho)^c) \\
&\quad \leqslant R^2(l^*(\zeta))^2 \exp[-\beta\gamma^{-1}[c(\Theta, \rho)\, L^{-3} \\
&\quad\quad - c(L+R)(\delta^* + \gamma\delta^{*-1} \log \delta^*\gamma^{-1}) - c(\zeta+\delta)]] \\
&\quad\quad + \exp[-\beta\gamma^{-1}[cR(c'\zeta^{3+q}\,\delta^2 - (\delta^* + \gamma(\delta^*)^{-1} \log(\delta^*\gamma^{-1}))) - 2C]]
\end{aligned} \quad (4.26)$$

It remains to choose the parameters R, δ, ζ, δ^* in such a way that the right hand side of (4.26) goes to zero. Making similar arguments as after (4.9), one can check that the choice $\delta = \zeta = \gamma^{(2(6+q))^{-1}}$, $R = c_1 \gamma^{-(5+q)/2(6+q)}$, c_1 a positive constant, and $\delta^* \approx \gamma^{1/2} \log \gamma^{-1/2}$ implies that, if $L \leqslant \gamma^{-\lambda}$ with $\lambda < (6(6+q))^{-1}$ then the right hand side of (4.26) goes to zero, and this ends the proof of the Theorem 2.5. ∎

5. ESTIMATES FOR THE INDEPENDENT MODEL

In this section we prove Theorem 2.2.

It is possible to compute an explicit expression for the density of ν_N. By using the integral representation of the δ-function on $\mathbb{R}^q$ we have

$$\begin{aligned} \frac{d\nu_N}{dm}(m) &= \int \prod_{i=1}^N \nu(d\sigma_i)\, \delta(m_N(\sigma) - m) \\ &= \left(\frac{N}{2\pi}\right)^q \int_{\mathbb{R}^q} d\lambda\, e^{-iN\lambda \cdot m} \left(\int \nu(dv)\, e^{i\lambda \cdot v} \right)^N \end{aligned} \tag{5.1}$$

We recall now the following integral representation of the Bessel functions (ref. 36, p. 47). For any $p \in \mathbb{C}$ such that $\mathrm{Re}(p) > -1/2$,

$$\mathcal{J}_p(z) = \frac{(z/2)^p}{\Gamma(p+1/2)\, \Gamma(1/2)} \int_0^\pi d\theta\, e^{iz\cos\theta} \sin^{2p}\theta, \qquad z \in \mathbb{C} \tag{5.2}$$

From the spherical symmetry of the problem, by using polar coordinates and recalling that $|S^{q-1}| = 2\pi^{q/2}/\Gamma(q/2)$, it is not difficult to obtain

$$\frac{d\nu_N}{dm}(m) = \frac{N^q \Gamma(q/2)}{\pi^{q/2} (N|m|)^{q-1}} \int_0^\infty dt \left(\frac{tN|m|}{2} \right)^{q/2} \mathcal{J}_{q/2-1}(tN|m|) \left(\frac{\mathcal{J}_{q/2-1}(t)}{(t/2)^{q/2-1}} \right)^N \tag{5.3}$$

We point out that $P_N^q(N|m|) \equiv N^{-q}(d\nu_N/dm)(m)$ is the well known formula of the density of the probability distribution for the Pearson's walk (ref. 36, p. 419). The integral in the r.h.s. of (5.3) is absolutely convergent for $N \geqslant 2$ and identically 0 for $|m| > 1$. So the density is well defined for $N \geqslant 2$ and identically 0 for $|m| > 1$.

Now we prove (2.20). Let $r \in (0, 1)$ and denote by B_r the closed ball in $\mathbb{R}^q$ of radius r and center in the origin. Fix $m \in B_r$ and let $h^* = h^*(m)$ be as in (2.16). Then

$$\frac{d\nu_N}{dm}(m) = N^q \exp[-NI(m)] \int \prod_{i=1}^N \nu_m(d\sigma_i)\, \delta\left(\sum_{i=1}^N \sigma_i - Nm \right) \tag{5.4}$$

having introduced the measure

$$\nu_m(dv) \equiv \phi(h^*)^{-1} e^{h^* \cdot v} \nu(dv) \tag{5.5}$$

Calling

$$\varphi_m(k) = e^{ik \cdot m} \int \nu_m(dv)\, e^{ik \cdot v}, \qquad k \in \mathbb{R}^q \tag{5.6}$$

and using again the integral representation of the δ-function, we get

$$\frac{d\nu_N}{dm}(m) = \exp[-NI(m)] \left(\frac{N}{2\pi}\right)^q \int dk\, \varphi_m(k)^N \tag{5.7}$$

so that

$$\varepsilon(m, N) = \frac{1}{N} \log\left[\left(\frac{N}{2\pi}\right)^q \int dk\, \varphi_m(k)^N\right] \tag{5.8}$$

Next we will prove the following lemma.

Lemma 5.1. Fix $r \in (0, 1)$ and let $\varphi_m(k)$ be as in (5.6) with $m \in B_r$. Then $k \mapsto \varphi_m(k)$ is a smooth complex function such that:

(i) $|\varphi_m(k)| \leqslant \varphi_m(0) = 1$ for any $k \in \mathbb{R}^q$.

(ii) In a neighbor of the origin we have the expansion

$$\varphi_m(k) = 1 - \tfrac{1}{2} Q_m(k) + \zeta_m(k), \qquad k \in \mathbb{R}^q \tag{5.9}$$

where $Q_m(\cdot)$ is a quadratic form, uniformly positive for $m \in B_r$, and $\zeta_m(k)$ a smooth function satisfying $|\zeta_m(k)| \leqslant c_1(r)\, |k|^3$ for some $c_1(r) > 0$.

(iii) There are positive constants $c_2(r)$ and $\kappa(r)$ such that

$$|\varphi_m(k)| \leqslant \frac{c_2(r)}{|k|^{q/2}}, \qquad \forall k \in \mathbb{R}^q\colon |k| \geqslant \kappa(r) \tag{5.10}$$

(iv) $|\varphi_m(k)|$ reaches its maximum value only for $k = 0$ which is a strict maximum for this function.

We fix $\delta \in (1/3, 1/2)$ and decompose

$$\int dk\, \varphi_m(k)^N = \sum_{i=1}^{3} G_i(N, \delta)$$

with

$$G_1(N, \delta) = \int_{|k| \leqslant N^{-\delta}} dk\, \varphi_m(k)^N$$
$$G_2(N, \delta) = \int_{N^{-\delta} < |k| \leqslant N} dk\, \varphi_m(k)^N \tag{5.11}$$
$$G_3(N, \delta) = \int_{|k| > N} dk\, \varphi_m(k)^N$$

We estimate the three terms separately. By changing variables and using Lemma 5.1, item (ii),

$$\begin{aligned} G_1(N, \delta) &= \frac{1}{N^{q/2}} \int_{|k| \leqslant N^{1/2-\delta}} dk\, \varphi_m\left(\frac{k}{\sqrt{N}}\right)^N \\ &= \frac{1}{N^{q/2}} \int_{|k| \leqslant N^{1/2-\delta}} dk \exp\left[N \log\left(1 - \frac{1}{2N} Q_m(k) + \zeta_m\left(\frac{k}{\sqrt{N}}\right)\right)\right] \\ &= \frac{1}{N^{q/2}} \int_{|k| \leqslant N^{1/2-\delta}} dk \exp\left[-\frac{1}{2} Q_m(k)\right] \left(1 + N\hat{\zeta}_m\left(\frac{k}{\sqrt{N}}\right)\right) \\ &= \frac{1}{N^{q/2}} \left(\frac{\pi}{\det Q_m}\right)^{q/2} [1 + O(N^{-(3\delta-1)})] \end{aligned} \tag{5.12}$$

where $\hat{\zeta}_m(k)$ is a smooth function with the same properties of $\zeta_m(k)$ so that the rest $O(N^{-(3\delta-1)})$ is uniformly bounded when $m \in B_r$. From items (ii), (iii) and (iv) of Lemma 5.1, there is $C_1(r) > 0$ such that for any N large enough $|\varphi_m(k)| \leqslant 1 - C_1(r)\, N^{2\delta}$ when $|k| \geqslant N^{-\delta}$. Then

$$|G_2(N, \delta)| \leqslant N^q\, |S^{q-1}| \sup_{|k| > N^{-\delta}} |\varphi_m(k)|^N \leqslant N^q\, |S^{q-1}| \left(1 - \frac{C_1(r)}{N^{2\delta}}\right)^N$$

so that, for some $C_2(r) > 0$,

$$|G_2(N, \delta)| \leqslant \exp[-C_2(r)\, N^{1-2\delta}] \tag{5.13}$$

Finally, by using item (iv) of Lemma 5.1,

$$|G_3(N, \delta)| \leqslant \int_{|k| > N} dk \left(\frac{c_2(r)}{|k|^{q/2}}\right)^N \leqslant N^q\, |S^{q-1}| \left(\frac{c_2(r)}{N^{q/2}}\right)^N \tag{5.14}$$

From (5.8), (5.12), (5.13) and (5.14) the bound (2.20) for some $c(r)>0$ follows.

We are left with the proof of (2.21). Denoting by P the probability distribution of σ, we have

$$\nu_N(|m|>r)=P(A_r) \tag{5.15}$$

where A_r is the cylinder set defined by

$$A_r=\left\{\sigma\colon \sum_{i,j=1}^{N} \sigma_i\cdot\sigma_j>r^2N^2\right\} \tag{5.16}$$

Consider now the sets

$$\Omega_{i,j}^{(r)}=\{\sigma\colon \sigma_i\cdot\sigma_j>r^2+r-1\},\qquad i,j=1,\ldots,N$$

and define the stochastic variable

$$\sigma\mapsto n(\sigma)\equiv \#\{(i,j)\colon \sigma\in\Omega_{i,j}^{(r)}\}$$

Clearly, for any σ,

$$\sum_{i,j=1}^{N}\sigma_i\cdot\sigma_j\leqslant (N^2-n(\sigma))(r^2+r-1)+n(\sigma)$$

so that, for any $r\in(0,1)$,

$$A_r\subseteq\{\sigma\colon n(\sigma)>f(r)\,N^2\},\qquad f(r)=\frac{r^2-(r^2+r-1)}{1-(r^2+r-1)}=\frac{1}{2+r} \tag{5.17}$$

Since $f([0,1])=[1/3,1/2]$, from (5.17) we conclude that, for any $r\in(0,1)$,

$$A_r\subseteq G_r,\qquad G_r\equiv\{\sigma\colon n(\sigma)>N^2/3\} \tag{5.18}$$

Then we need an estimate of the probability of G_r. For any subset I of $\{1,\ldots,N\}$ let

$$\Omega_I^{(r)}\equiv\{\sigma\colon \text{there are } |I| \text{ indexes } j\in\{1,\ldots,N\}\backslash I \text{ such that } \sigma\in\Omega_{i,j}^{(r)}\}$$

We claim that

$$G_r\subseteq\bigcup_{I\colon |I|\geqslant N/6}\Omega_I^{(r)} \tag{5.19}$$

In fact, it is possible to extract at least $N/6$ disjoint pairs (i, j) from a set of pairs whose cardinality is bigger than $N^2/3$. Now, since the σ's are i.i.d., for any I,

$$P(\Omega_I^{(r)}) = P\left(\bigcap_{i=1}^{|I|} \Omega_{i,\,i+|I|}^{(r)}\right) = P(\Omega_{1,2}^{(r)})^{|I|} \tag{5.20}$$

On the other hand, by using polar coordinates,

$$P(\Omega_{1,2}^{(r)}) = \int \nu(d\sigma_1) \int \nu(d\sigma_2)\, \mathbb{1}_{\{\sigma_1 \cdot \sigma_2 > r^2 + r - 1\}} \leqslant \frac{\arccos(r^2 + r - 1)}{\pi} \tag{5.21}$$

Collecting together (5.18), (5.19),(5.20), (5.21) and observing that

$$0 < \frac{\arccos(r^2 + r - 1)}{\pi} < \frac{\sqrt{6(1-r)}}{\pi}, \qquad \text{for any} \quad r \in (0, 1)$$

we finally get

$$P(A_r) \leqslant N \binom{N}{N/6} P(\Omega_{1,2}^{(r)})^{N/6} \leqslant N \exp[N I_{\mathrm{Ber}}(1/6)] \left(\frac{\sqrt{6(1-r)}}{\pi}\right)^{N/6} \tag{5.22}$$

where $I_{\mathrm{Ber}}(\cdot)$ is the Bernoulli entropy. Clearly (5.22) gives (2.21) for a suitable choice of $b > 0$. Theorem 2.2 is proved. ∎

Proof of Lemma 5.1. Since ν_m is compactly supported, the smoothness of $\varphi_m(k)$ comes from direct inspection. Now we prove properties (i)–(iv).

(i) It holds trivially.

(ii) From the definition of h^*,

$$\int \nu_m(dv)\, v = \phi(h^*)^{-1} \int \nu(dv)\, e^{h^* \cdot v} v = (\nabla_h \log \phi)(h^*) = m \tag{5.23}$$

Then, by expanding φ_m in $k = 0$, we get (5.9) with

$$Q_m(k) = \int \nu_m(dv)[k \cdot (v - m)]^2, \qquad \zeta_m(k) = \int \nu_m(dv)\, e^{i\eta_{m,k}(v)}[k \cdot (v - m)]^3 \tag{5.24}$$

where $\eta_{m,k}(v)$ is a suitable number in the interval $[0 \wedge k \cdot (v - m), 0 \vee k \cdot (v - m)]$. Recalling the definition (5.5) of ν_m and (2.16) of h^* it is easy to check that $Q_m(k)$ and $\zeta_m(k)$ have the desired properties stated in the lemma. We omit the details.

(iii) We analyze separately the cases $q=2$ and $q=3$:

($q=2$) Calling α the angle between the vectors k and m and using polar coordinates, we have, for any $|k|>0$,

$$\varphi_m(k)=\frac{e^{-i\,|k|\,|m|\cos\alpha}}{2\pi\mathscr{I}_0(it^*)}\int_0^{2\pi}d\theta\,e^{t^*\cos\theta+i\,|k|\cos(\theta-\alpha)} \tag{5.25}$$

where we used (2.14) and that $|h^*|=t^*$, see (2.16) and (2.17). Now we recall the following integral representation of Bessel functions of integer order n (ref. 36, p. 19),

$$\mathscr{I}_n(z)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{iz\sin\theta-in\theta},\qquad z\in\mathbb{C} \tag{5.26}$$

which gives the Fourier expansion

$$e^{iz\sin\phi}=\sum_{n=-\infty}^{+\infty}\mathscr{I}_n(z)\,e^{in\phi},\qquad z\in\mathbb{C},\quad \phi\in\mathbb{R}$$

so that

$$\varphi_m(k)=\frac{e^{-i\,|k|\,|m|\cos\alpha}}{\mathscr{I}_0(it^*)}\sum_{n=-\infty}^{+\infty}\mathscr{I}_n(|k|)\,e^{in(\pi/2-\alpha)}\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{t^*\cos\theta+on\theta} \tag{5.27}$$

Using again (5.26) we get

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{t^*\cos\theta+in(\theta+\pi/2)}=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{t^*\sin\theta-in\theta}=\mathscr{I}_n(it^*) \tag{5.28}$$

and so

$$|\varphi_m(k)|\leqslant\frac{1}{\mathscr{I}_0(it^*)}\sum_{n=-\infty}^{+\infty}|\mathscr{I}_n(|k|)|\,|\mathscr{I}_n(it^*)| \tag{5.29}$$

In ref. 36, p. 205–206 it is proven that, for any x and p positive,

$$\mathscr{I}_{\pm p}(x)=\sqrt{\frac{2}{\pi x}}\left[\cos\left(x\mp\frac{p\pi}{2}-\frac{\pi}{4}\right)Q_+(x,p)-\sin\left(x\mp\frac{p\pi}{2}-\frac{\pi}{4}\right)Q_-(x,p)\right] \tag{5.30}$$

where

$$Q_{\pm}(x, p) \equiv \frac{1}{2\Gamma(p+1/2)} \int_0^\infty du\, e^{-u} u^{p-1/2} \left[\left(1 + \frac{iu}{2x}\right)^{p-1/2} \pm \left(1 - \frac{iu}{2x}\right)^{p-1/2} \right] \tag{5.31}$$

Now

$$|Q_{\pm}(x, 0)| \leqslant 1 \tag{5.32}$$

and, for any integer $n \geqslant 1$,

$$\begin{aligned} |Q_{\pm}(x, n)| &\leqslant \frac{1}{\Gamma(n+1/2)} \int_0^\infty du\, e^{-u} u^{1-1/2} \left[\left(1 + \left(\frac{u}{2x}\right)^2\right)^{n/2-1/4} \right] \\ &\leqslant \frac{1}{\Gamma(n+1/2)} \left[2^{n/2-1/4} \int_0^{2x} du\, e^{-u} u^{n-1/2} \right. \\ &\quad \left. + \frac{2^{n/2-1/4}}{(2x)^{1-1/2}} \int_{2x}^\infty du\, e^{-u} u^{2n-1} \right] \\ &\leqslant 2^{n/2-1/4} \left[1 + \frac{\Gamma(2n)}{(2x)^{n-1/2}\, \Gamma(n+1/2)} \right] \\ &= 2^{n/2-1/4} \left[1 + \frac{2^{n-1/2}\Gamma(n)}{\sqrt{\pi}\, x^{n-1/2}} \right] \end{aligned} \tag{5.33}$$

where in the last equality we used the "Legendre's duplication formula"

$$2^{2z-1}\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2z), \qquad z \in \mathbb{C}$$

In ref. 36, p. 49 it is proven also that for any real order $p > -\frac{1}{2}$,

$$|\mathscr{I}_p(z)| \leqslant \frac{(|z|/2)^p}{\Gamma(p+1)} \exp[\mathrm{Im}(z)], \qquad z \in \mathbb{C} \tag{5.34}$$

We use (5.30), (5.32) and (5.33) to bound $|\mathscr{I}_n(|k|)|$ and, since $\mathscr{I}_{-n}(z) = (-1)^n \mathscr{I}_n(z)$ for any integer n, we can use (5.34) to bound $|\mathscr{I}_n(-it^*)|$ for any $n \in \mathbb{Z}$. From (5.29) we get

$$|\varphi_m(k)| \leqslant \sqrt{\frac{2}{\pi\,|k|}}\left[1+\frac{4e^{-t^*}}{\mathscr{I}_0(it^*)}\sum_{n\geqslant 1}\left(\frac{t^*}{2}\right)^n \frac{2^{1/2-1/4}}{\Gamma(n+1)}\left(1+\frac{2^{n-1/2}\Gamma(n)}{\sqrt{\pi}\,|k|^{n-1/2}}\right)\right]$$

$$\leqslant \sqrt{\frac{2}{\pi\,|k|}}\left[1+\frac{4}{\mathscr{I}_0(it^*)}+\frac{2\sqrt{2}\,e^{-t^*}}{\mathscr{I}_0(it^*)}\sum_{n\geqslant 1}\frac{(\sqrt{2}\,t^*)^n}{\sqrt{n}\,|k|^{n-1/2}}\right]$$

$$\leqslant \sqrt{\frac{2}{\pi\,|k|}}\left(1+\frac{4}{\mathscr{I}_0(it^*)}\right)+\frac{4e^{-t^*}}{\pi\mathscr{I}_0(it^*)}\sum_{n\geqslant 1}\left(\frac{\sqrt{2}\,t^*}{|k|}\right)^n \tag{5.35}$$

Since t^* is an increasing function of $|m|$, by choosing $\kappa(r)>\sqrt{2}\,t^*(r)$, from (5.35) we get (5.10) for $q=2$ with $c_2(r)$ a suitable function of $t^*(r)$.

($q=3$) As before, calling α the angle between the vectors k and m, by introducing polar coordinates and recalling (2.14), we have, for any $|k|>0$,

$$\varphi_m(k)=\frac{t^* e^{-i\,|k|\,|m|\cos\alpha}}{4\pi\sinh(t^*)}\int_0^\pi d\theta$$

$$\times\int_0^{2\pi} d\psi\; e^{t^*\cos\theta+i\,|k|\cos\alpha\cos\theta} e^{i\,|k|\sin\alpha\sin\theta\cos\psi}\sin\theta$$

$$=\frac{t^* e^{-i\,|k|\,|m|\cos\alpha}}{2\sinh(t^*)}\int_0^\pi d\theta\; e^{t^*\cos\theta+i\,|k|\cos\alpha\cos\theta}\mathscr{I}_0(|k|\sin\alpha\sin\theta)\sin\theta \tag{5.36}$$

By Fourier–Legendre expansion of $\exp[t^*\cos\theta]$, (5.36) becomes

$$\varphi_m(k)=\frac{t^* e^{-i\,|k|\,|m|\cos\alpha}}{2\sinh(t^*)}\sum_{n\geqslant 0} c_n(t^*)$$

$$\times\int_0^\pi d\theta\; e^{i\,|k|\cos\alpha\cos\theta}\mathscr{I}_0(|k|\sin\alpha\sin\theta)\,P_n(\cos\theta)\sin\theta \tag{5.37}$$

where $P_n(x)$, $x\in[-1,1]$, is the Legendre polynomial of order n and

$$c_n(t^*)=\frac{2n+1}{2}\int_0^\pi d\theta\; e^{t^*\cos\theta}P_n(\cos\theta)\sin\theta \tag{5.38}$$

By applying the Gegenbauer's formulae (ref. 36, p. 50, 379),

$$\int_0^\pi d\theta\; e^{iz\cos\theta}P_n(\cos\theta)\sin\theta = i^n\sqrt{\frac{2\pi}{z}}\,\mathscr{I}_{n+1/2}(z), \qquad z\in\mathbb{C}$$

and

$$\int_0^{\pi} d\theta\, e^{iz\cos\alpha\cos\theta} \mathscr{I}_0(z\sin\alpha\sin\theta)\sin\theta P_n(\cos\theta)$$

$$= \sqrt{\frac{2\pi}{z}}\, i^n \mathscr{I}_{n+1/2}(z)\, P_n(\cos\alpha), \qquad z\in\mathbb{C}$$

to (5.38) and (5.37) respectively, we finally obtain the estimate

$$|\varphi_m(k)| \leqslant \frac{\pi}{\sinh(t^*)}\sqrt{\frac{t^*}{|k|}}\sum_{n\geqslant 0}(2n+1)\,|\mathscr{I}_{n+1/2}(-it^*)|\;|\mathscr{I}_{n+1/2}(|k|)|$$

where we used also that $|P_n(x)|\leqslant 1$ for any $x\in[-1,1]$ and $n\in\mathbb{N}$. Now we proceed analogously to the case $q=2$. We uses (5.34) to bound $|\mathscr{I}_{n+1/2}(-it^*)|$ and the integral representation (5.30) to get a bound for $|\mathscr{I}_{n+1/2}(|k|)|$ with the gain of a factor $|k|^{-1/2}$ (observe that, by reasoning as in (5.33), for any $n\geqslant 0$ and $x>0$, $|Q_\pm(x,n+1/2)|\leqslant 2^{n/2}[1+\Gamma(2n+1)/((2x)^n\,\Gamma(n+1))]$). We omit the details.

(iv) From (i) and (ii), $k=0$ is a strict absolute maximum. From (5.25) and (5.36) when $q=2$ and $q=3$ respectively, $|\varphi_m(k)|=|\Phi_\alpha(|k|)|$ where, for any $\alpha\in[0,\pi]$, $\Phi_\alpha(t)$, $t\in\mathbb{R}$, is the characteristic function of a suitable real random variable ξ_α. By standard results on characteristic functions (ref. 16, p. 501, Lemma 4), if for some α there is $\lambda>0$ such that $|\Phi_\alpha(\lambda)|=1$, then $|\Phi_\alpha(t)|$ is a periodic function of period λ. But this implies

$$\limsup_{|k|\to+\infty}|\varphi_m(|k|)|=1$$

which contradicts property (iii). ∎

REFERENCES

1. G. Alberti, G. Bellettini, M. Cassandro, and E. Presutti, Surface tension in Ising system with Kac potentials, *J. Stat. Phys.* **82**(3–4):743–796 (1996).
2. O. Benois, T. Bodineau, P. Buttà, and E. Presutti, On the validity of the van der Waals theory of surface tension, *Markov Proc. Rel. Fields* **3**(2):175–198 (1997).
3. T. Bodineau, Interface in a one-dimensional Ising spin system, *Stoch. Proc. Appl.* **61**:1–23 (1996).
4. T. Bodineau, Interface for one-dimensional random Kac potentials, *Ann. Inst. H. Poincaré* **33**(5):559–590 (1997).

5. T. Bodineau and E. Presutti, Phase diagram of Ising systems with additional long range forces, *Commun. Math. Phys.* **189**:287–298 (1997).
6. A. Bovier, V. Gayrard, and P. Picco, Large deviation principles for the Hopfield model and the Kac–Hopfield model, *Prob. Theor. Rel. Fields* **101**:511–546 (1995).
7. A. Bovier, V. Gayrard, and P. Picco, Distribution of overlap profiles in the one-dimensional Kac–Hopfield model, *Commun. Math. Phys.* **186**:323–379 (1997).
8. A. Bovier and M. Zahradnik, The low temperature phase of Kac–Ising models, *J. Stat. Phys.* **87**:311–332 (1997).
9. J. Bricmont, J. Fontaine, and J. Landau, On the uniqueness of the equilibrium state for plane rotator, *Commun. Math. Phys.* **56**:281–286 (1977).
10. P. Buttà, I. Merola, and E. Presutti, On the validity of the van der Waals theory in Ising systems with long range interactions, *Markov Proc. Rel. Fields* **3**(1):63–88 (1997).
11. M. Cassandro, R. Marra, and E. Presutti, Corrections to the critical temperature in 2d Ising systems with Kac potentials, *J. Stat. Phys.* **78**:1131–1138 (1995).
12. M. Cassandro, E. Orlandi, and E. Presutti, Interfaces and typical Gibbs configurations for one-dimensional Kac potentials, *Prob. Theor. Rel. Fields* **96**:57–96 (1993).
13. M. Cassandro and E. Presutti, Phase transitions in Ising systems with long but finite range, *Markov Proc. Rel. Fields* **2**:241–262 (1996).
14. A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, Glauber evolution with Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics, *Nonlinearity* **7**:1–67 (1994).
15. R. L. Dobrushin and S. Shlosman, Absence of breakdown of continuous symmetry in two dimensional models of statistical physics, *Commun. Math. Phys.* **42**:31–40 (1975).
16. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II (Wiley, New York/London, 1963).
17. J. Fröhlich and C. E. Pfister, On the absence of spontaneous symmetry breaking and crystalline ordering in two dimensional systems, *Commun. Math. Phys.* **81**:277–298 (1981).
18. J. Fröhlich and C. E. Pfister, Spin waves, vortices, and the structure of equilibrium states in the classical XY model, *Commun. Math. Phys.* **89**:303–327 (1983).
19. J. Fröhlich, B. Simon, and T. Spencer, Infrared bound, phase transitions and continuous symmetry breaking, *Commun. Math. Phys.* **50**:79–85 (1976).
20. J. Fröhlich and T. Spencer, The Kosterlitz–Thouless transition in two dimensional abelian spin systems and the Coulomb gas, *Comm. Math. Phys.* **81**:527–607 (1981).
21. H. O. Georgii, *Gibbs Measures and Phase Transitions*, De Gruyter Studies in Mathematics, Vol. 9 (Walter de Gruyter, Berlin/New York, 1988).
22. P. C. Hemmer and J. L. Lebowitz, Systems with weak long-range potentials, in "*Phase Transition and Critical Phenomena*," Vol. 5b, Domb and Green, eds. (1976).
23. K. R. Ito, Clustering in low-dimensional $SO(n)$-invariant statistical models with long range interactions, *J. Stat. Phys.* **29**:747–760 (1982).
24. M. Kac, G. Uhlenbeck, and P. C. Hemmer, On the van der Waals theory of vapour-liquid equilibrium. I. Discussion of a one-dimensional model, *J. Math. Phys.* **4**:216–228 (1963).
25. M. Kac, G. Uhlenbeck, and P. C. Hemmer, On the van der Waals theory of vapour-liquid equilibrium. II. Discussion of the distribution functions, *J. Math. Phys.* **4**:229–247 (1963).
26. M. Kac, G. Uhlenbeck, and P. C. Hemmer, On the van der Waals theory of vapour-liquid equilibrium. III. Discussion of the critical region, *J. Math. Phys.* **5**:60–74 (1964).
27. H. Kesten and R. H. Schonmann, Behavior in large dimensions of the Potts and Heisenberg models, *Rev. Math. Phys.* **1**:147–182 (1990).
28. J. M. Kosterlitz and D. V. Thouless, Ordering metastability and phase transitions in twodimensional systems, *J. Phys. C* **6**:1181–1203 (1973).
29. J. Lebowitz and O. Penrose, Rigorous treatment of the Van der Waals Maxwell theory of the liquid vapour transition, *J. Math. Phys.* **7**:98–113 (1966).

30. A. Messager, S. Miracle, and C. E. Pfister, Correlation inequalities and uniqueness of equilibrium state for the planar rotator ferromagnetic model, *Commun. Math. Phys.* **58**:19–29 (1978).
31. C. E. Pfister, On the symmetry of Gibbs states in two dimensional lattice systems, *Commun. Math. Phys.* **79**:181–188 (1981).
32. D. Ruelle, Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.* **18**:127 (1970).
33. D. Ruelle, Probability estimates for continuous spin systems, *Commun. Math. Phys.* **50**:189–194 (1976).
34. S. B. Shlosman, Decrease of correlations in two-dimensional models with continuous symmetry group, *Theor. Math. Phys.* **37**:1118–1120 (1978).
35. C. J. Thompson and M. Silver, The classical limit of *n*-vector spin models, *Commun. Math. Phys.* **33**:53–60 (1973).
36. G. N. Watson, *Theory of Bessel Functions* (Cambridge U.P., London, 1944).